# Hadronic light-by-light contribution to $(g-2)_\mu$: a dispersive approach

Gilberto Colangelo

UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

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Based on:

JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 ..................................................... +B. Kubis

# Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
- Gauge invariance and crossing symmetry
- Master Formula

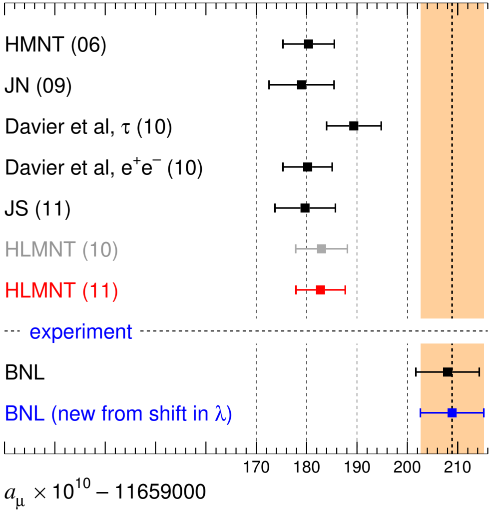
A dispersion relation for HLbL

Numerics
- - Pion box contribution
- - Pion rescattering contribution

Outlook and Conclusions

# Status of $(g-2)_\mu$, experiment vs SM

Hagiwara et al. 2012



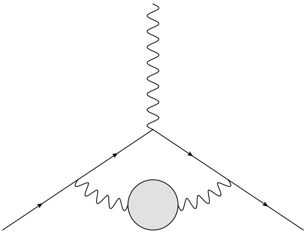
# Status of $(g-2)_\mu$, experiment vs SM

| | $a_\mu\,[10^{-11}]$ | $\Delta a_\mu\,[10^{-11}]$ |
|---|---|---|
| experiment | 116 592 089. | 63. |
| QED $\mathcal{O}(\alpha)$ | 116 140 973.21 | 0.03 |
| QED $\mathcal{O}(\alpha^2)$ | 413 217.63 | 0.01 |
| QED $\mathcal{O}(\alpha^3)$ | 30 141.90 | 0.00 |
| QED $\mathcal{O}(\alpha^4)$ | 381.01 | 0.02 |
| QED $\mathcal{O}(\alpha^5)$ | 5.09 | 0.01 |
| QED total | 116 584 718.95 | 0.04 |
| electroweak, total | 153.6 | 1.0 |
| HVP (LO) [Hagiwara et al. 11] | 6 949. | 43. |
| HVP (NLO) [Hagiwara et al. 11] | −98. | 1. |
| HLbL [Jegerlehner-Nyffeler 09] | 116. | 40. |
| HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14] | 12.4 | 0.1 |
| HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14] | 3. | 2. |
| theory | 116 591 855. | 59. |

# Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved

→ talk by Lehner



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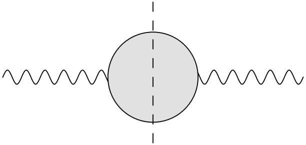
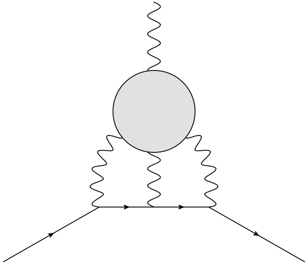


- basic principles: unitarity and analyticity
- direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- dedicated $e^+e^-$ program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)
- alternative approach: lattice

→ talk by Lehner

# Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
  $\rightarrow$ talk by Lehner
- Hadronic light-by-light (HLbL) is more problematic:



- 4-point fct. of em currents in QCD
- "*it cannot be expressed in terms of measurable quantities*"
- up to now, only model calculations
- lattice QCD is making fast progress

RBC/UKQCD, Mainz, several parallel talks

# Analytic Approaches to Hadronic light-by-light

- Model calculations
  - ENJL — Bijnens, Pallante, Prades (95-96)
  - NJL and hidden gauge — Hayakawa, Kinoshita, Sanda (95-96)
  - nonlocal $\chi$QM — Dorokhov, Broniowski (08)
  - AdS/CFT — Cappiello, Cata, D'Ambrosio (10)
  - Dyson-Schwinger — Goecke, Fischer, Williams (11)
  - constituent $\chi$QM — Greynat, de Rafael (12)
  - resonances in the narrow-width limit — Pauk, Vanderhaeghen (14)
- Impact of rigorously derived constraints
  - high-energy constraints taken into account in several models above
    addressed specifically by — Knecht, Nyffeler (01)
  - high-energy constraints related to the axial anomaly — Melnikov, Vainshtein (04) and Nyffeler (09)
  - sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen (12)
    see also: workshop MesonNet (13)
  - low-energy constraints–pion polarizabilities — Engel, Ramsey-Musolf (13)

# Different evaluations of HLbL

Jegerlehner-Nyffeler 2009

| Contribution | BPaP(96) | HKS(96) | KnN(02) | MV(04) | BP(07) | PdRV(09) | N/JN(09) |
|---|---|---|---|---|---|---|---|
| $\pi^0, \eta, \eta'$ | 85±13 | 82.7±6.4 | 83±12 | 114±10 | – | 114±13 | 99±16 |
| $\pi$, $K$ loops | −19±13 | −4.5±8.1 | – | – | – | −19±19 | −19±13 |
| " " + subl. in $N_c$ | – | – | – | 0±10 | – | – | – |
| axial vectors | 2.5±1.0 | 1.7±1.7 | – | 22± 5 | – | 15±10 | 22± 5 |
| scalars | −6.8±2.0 | – | – | – | – | −7± 7 | −7± 2 |
| quark loops | 21± 3 | 9.7±11.1 | – | – | – | 2.3 | 21± 3 |
| total | 83±32 | 89.6±15.4 | 80±40 | 136±25 | 110±40 | 105±26 | 116±39 |

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*$K$s are subdominant*)
- heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

# Recent progress on the lattice

- $a_\mu^{\mathrm{HLbL}}$ in finite-volume QCD and QED ($\mathrm{QED_L}$): talks by Blum and Izubuchi
  - (RBC/UKQCD): Connected diagram with $m_\pi = 171$ MeV; $a_\mu^{\mathrm{HLbL}} = 13.21(68) \times 10^{-10}$ PRD93(2016)014503
  - (RBC/UKQCD): Connected and leading disconnected diagram with $m_\pi = 139$ MeV; $a_\mu^{\mathrm{HLbL}} = 5.35(1.35) \times 10^{-10}$
    (potentially large finite-volume systematics) PRL118(2017)022005
- $a_\mu^{\mathrm{HLbL}}$ in finite-volume QCD and infinite-volume QED: talk by Asmussen
  - (Mainz) Method proposed and successfully tested against the lepton-loop analytic result arXiv:1510.08384 arXiv:1609.08454
  - (RBC/UKQCD) Similar method plus subtraction scheme to reduce systematic errors; successfully tested against lepton-loop analytic result arXiv:1705.01067
- The pion pole contribution to $a_\mu^{\mathrm{HLbL}}$: talk by Gerardin
  - (Mainz) Calculation of the pion transition form factor on the lattice gives: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = 6.50(83) \times 10^{-10}$ PRD94(2016)074507

# Our approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

- model independent $\Rightarrow$ rely on dispersion relations
- as data-driven as possible
- takes into account high-energy constraints
  [OPE, perturbative QCD]
  (work in progress, not discussed here)

Alternative dispersive approach for the $\mu$-FF Pauk-Vanderhaeghen (14)

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# Hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle = \left( q_\mu q_\nu - g_{\mu\nu} q^2 \right) \Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^\infty dt \frac{\mathrm{Im}\bar{\Pi}(t)}{t(t-q^2)}$$

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Easy!

# The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent

Eichmann *et al.* (14)

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

# Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
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- of these 54 only 7 are distinct structures

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta} q_{1\alpha} q_{2\beta} q_{3\gamma} q_{4\delta}, \\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right), \\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right), \\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right), \\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right), \\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\left(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right)\right. \\
&\quad \left. + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\right), \\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma\left(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right)\right. \\
&\quad \left. + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu}\right)\right) \\
&\quad - q_4^\lambda\left(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right)\right. \\
&\quad \left. + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma}\right)\right) \\
&\quad + q_3\cdot q_4\left(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\right).
\end{aligned}
$$

# Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

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- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions $\Pi_i$ are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

# HLbL contribution to $a_\mu$

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2) = -k^\rho \frac{\partial}{\partial k^\sigma}\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2).$$

Contribution to $a_\mu$: $m := m_\mu$

$$a_\mu = \frac{-1}{48m}\mathrm{Tr}\Big\{(\not{p}+m)[\gamma^\rho,\gamma^\sigma](\not{p}+m)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p)\Big\}$$

$$\Gamma_{\rho\sigma} = e^6\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2(q_1+q_2)^2}\frac{\gamma^\mu(\not{p}+\not{q}_1+m)\gamma^\lambda(\not{p}-\not{q}_2+m)\gamma^\nu}{((p+q_1)^2-m^2)((p-q_2)^2-m^2)}\times$$

$$\times\frac{\partial}{\partial k^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2)\Big|_{k=0}$$

BTT basis (no kin. singularities!) $\Rightarrow$ limit $k_\mu \to 0$ unproblematic

# Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the $\Pi_i$
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

# Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where $Q_i^\mu$ are the Wick-rotated four-momenta and $\tau$ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

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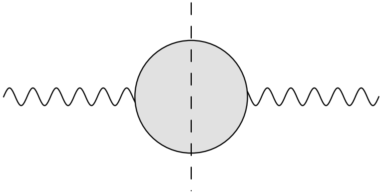
# Setting up the dispersive calculation

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

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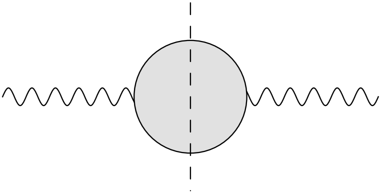


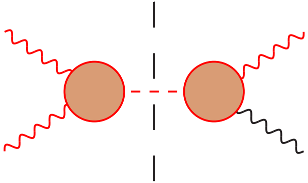
$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \text{hadrons})$$

For HLbL things are more complicated

# Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = $\delta$-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
Input: pion transition form factor Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)
First results of direct lattice calculations Gerardin, Mayer, Nyffeler (2016)

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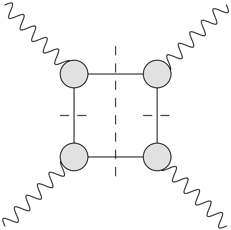
$\pi$-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole

– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

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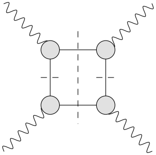
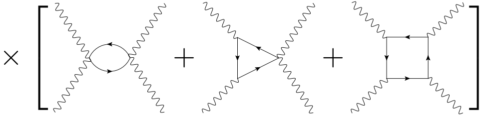
$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



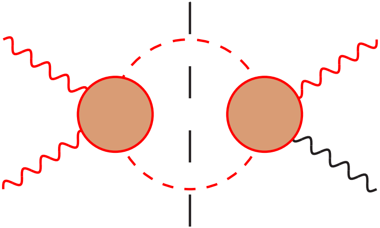
$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$



# Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



The "rest" with $2\pi$ intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

# Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will not be discussed here

# Partial wave expansion for $2\pi$ contributions

To complete the program of writing down a dispersion relation for two-pion contributions **is not easy**:

- unitarity relations are diagonal in a helicity amplitude basis;
- the helicity basis relevant for $(g-2)_\mu$ is the one with one on-shell photon, which has 27 elements;
- in the limit $q_4^2, q_4^\sigma \to 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- there is freedom in the choice of this subset (singly-on-shell basis);
- the arbitrariness in the choice of the 27 elements of the singly-on-shell basis does not influence the final result because of sum rules
- these sum rules follow from the assumption that the HLbL tensor has a uniform behaviour at short distances

# $S$-wave $2\pi$ contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s'\,\mathrm{Im}h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im}h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t'\,\mathrm{Im}h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im}h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u'\,\mathrm{Im}h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im}h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im}h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im}h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im}h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im}h^0_{00,++}(s')\Big)$$

Analogous expressions for the $D$, $G$ and all higher waves have been derived but are too long to be shown

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- Pion rescattering contribution

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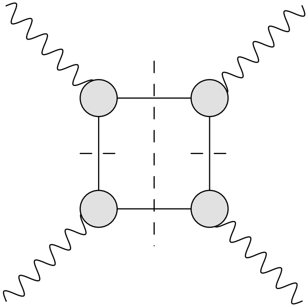
# Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



# Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

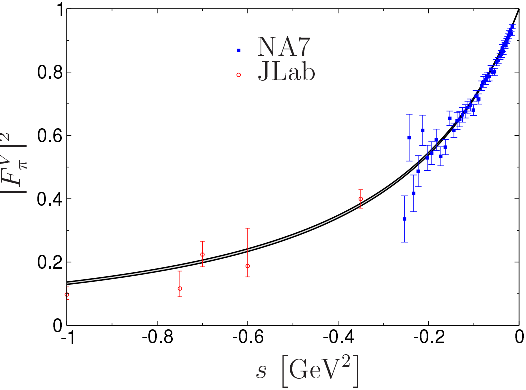
where

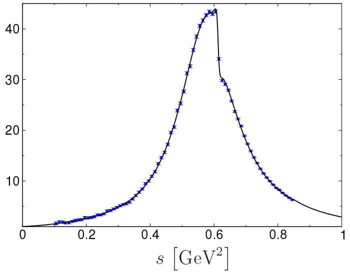
$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

# Pion box contribution




Uncertainties are negligibly small:

$$a_\mu^{\mathrm{FsQED}} = -15.9(2) \cdot 10^{-11}$$

# Pion box contribution

| Contribution | BPaP(96) | HKS(96) | KnN(02) | MV(04) | BP(07) | PdRV(09) | N/JN(09) |
|---|---|---|---|---|---|---|---|
| $\pi^0, \eta, \eta'$ | 85±13 | 82.7±6.4 | 83±12 | 114±10 | – | 114±13 | 99±16 |
| $\pi, K$ loops | −19±13 | −4.5±8.1 | – | – | – | −19±19 | −19±13 |
| " " + subl. in $N_c$ | – | – | – | 0±10 | – | – | – |
| axial vectors | 2.5±1.0 | 1.7±1.7 | – | 22± 5 | – | 15±10 | 22± 5 |
| scalars | −6.8±2.0 | – | – | – | – | −7± 7 | −7± 2 |
| quark loops | 21± 3 | 9.7±11.1 | – | – | – | 2.3 | 21± 3 |
| total | 83±32 | 89.6±15.4 | 80±40 | 136±25 | 110±40 | 105±26 | 116±39 |

Uncertainties are negligibly small:

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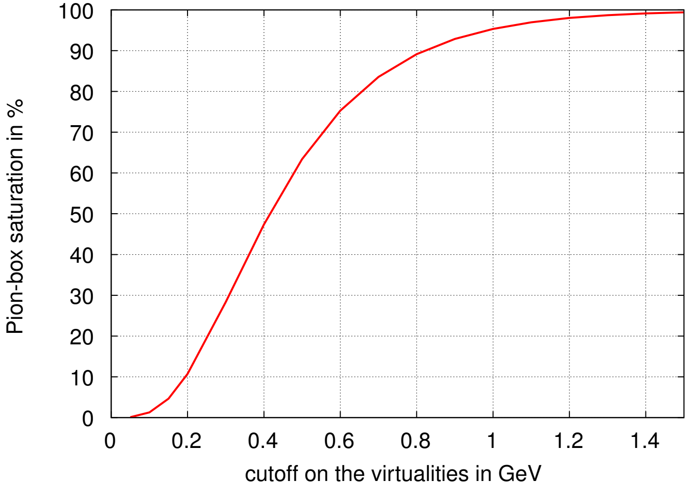
# Pion-box saturation with photon virtualities



# Check of the partial-wave formalism

Comparison partial-wave expansion of the pion-box vs. full result

| $J_{\text{max}}$ | $\delta_{J_{\text{max}}}$ | $\Delta_{J_{\text{max}}}$ |
|---|---|---|
| 0 | 29.2% | 55.4% |
| 2 | 10.4% | 20.9% |
| 4 | 4.3% | 11.0% |
| 6 | 2.4% | 6.2% |
| 8 | 1.5% | 3.7% |
| 10 | 1.0% | 2.4% |
| 12 | 0.7% | 1.6% |
| 14 | 0.6% | 1.1% |

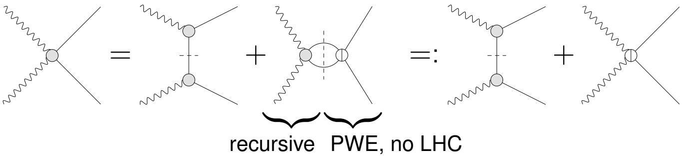
where

$$\delta_{J_{\text{max}}} := 1 - \frac{a_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}} \qquad \Delta_{J_{\text{max}}} := \frac{\left|\mathbf{a}_{\mu,J_{\text{max}}}^{\pi\text{-box, PW}} - \mathbf{a}_\mu^{\pi\text{-box}}\right|}{\left|\mathbf{a}_\mu^{\pi\text{-box}}\right|}$$

Convergence for real helicity amplitudes should be much better

# First evaluation of *S*- wave $2\pi$-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



# First evaluation of *S*- wave $2\pi$-rescattering

Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to $\infty$]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

*S*-wave contributions: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

$a_\mu^{\text{HLbL}}$ in $10^{-11}$ units

| cutoff | 1 GeV | 1.5 GeV | 2 GeV | $\infty$ |
|---|---|---|---|---|
| $I = 0$ | −9.2 | −9.5 | −9.3 | −8.8 |
| $I = 2$ | 2.0 | 1.3 | 1.1 | 0.9 |
| sum | −7.3 | −8.3 | −8.3 | −7.9 |

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Recall $\pi$-Box: $a_\mu^{\pi-\text{box}} = -15.9(2) \cdot 10^{-11}$

# First evaluation of $S$- wave $2\pi$-rescattering

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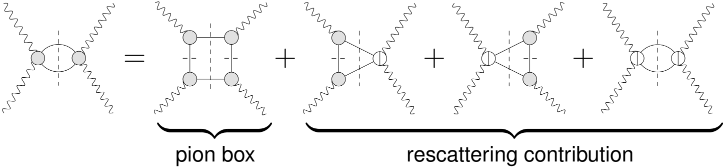
Recall $\pi$-Box: $a_\mu^{\pi-\text{box}} = -15.9(2) \cdot 10^{-11}$

# Our first numerical result

Two-pion contributions to HLbL:



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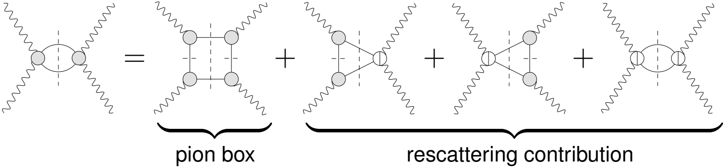


$$a_\mu^{\pi\text{-box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

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# Conclusions

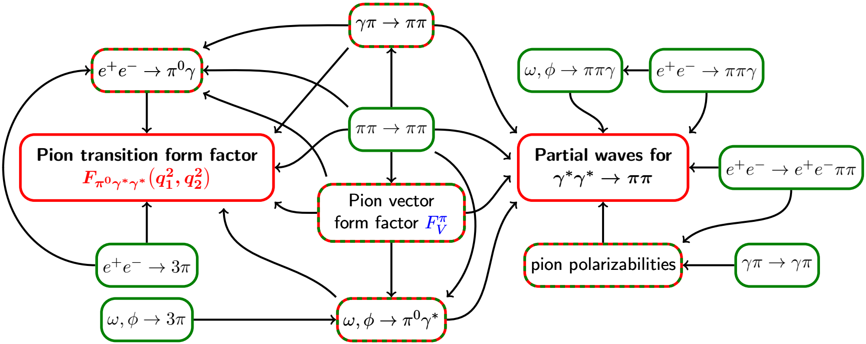
- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to $a_\mu$ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes
- I have presented results for the pion-box and the *S*-wave pion-rescattering contributions:
  model independence = much reduced uncertainties

# Outlook

- More work is needed to complete the evaluation of contributions of $2\pi$ intermediate states
  - take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
  - estimate the dependence on the $q^2$ of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
  - $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- short-distance constraints need to be derived and imposed

# Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)




Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists

# Backup Slides

# Detour: the subprocess $\gamma^*\gamma^* \to \pi\pi$

Consider $\gamma^*(q_1, \lambda_1)\gamma^*(q_2, \lambda_2) \to \pi^a(p_1)\pi^b(p_2)$:

$$W_{ab}^{\mu\nu}(p_1, p_2, q_1) = i \int d^4x\, e^{-iq_1\cdot x} \langle \pi^a(p_1)\pi^b(p_2)| T\{j_{\rm em}^\mu(x) j_{\rm em}^\nu(0)\} |0\rangle$$

General tensor decomposition ($q_i$, $i = 1, \ldots, 3$, $q_3 = p_2 - p_1$):

$$W^{\mu\nu} = g^{\mu\nu} W_1 + \sum_{i,j} q_i^\mu q_j^\nu W_2^{ij}$$

gives ten independent scalar functions.
Gauge invariance requires:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

# Gauge invariance: Bardeen-Tung-Tarrach approach

Consider the projector

Bardeen, Tung (68)

$$I^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

which satisfies

$$I_\mu{}^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}, \qquad q_1^\mu I_{\mu\nu} = q_2^\nu I_{\mu\nu} = 0$$

and contract it twice with $W_{\mu\nu}$, leaving it invariant:

$$W_{\mu\nu} = I_{\mu\mu'} I_{\nu'\nu} W^{\mu'\nu'} = \sum_{i=1}^{5} \bar{T}^i_{\mu\nu} \bar{A}_i = \sum_{i=1}^{5} T^i_{\mu\nu} A_i$$

The $\bar{A}_i$ are free of kinematic singularities, but have zeros. To remove the zeros from the $\bar{A}_i$ $\Rightarrow$ remove the poles from the $\bar{T}_i^{\mu\nu}$

# Gauge invariance: Bardeen-Tung-Tarrach approach

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$
$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$
$$T_3^{\mu\nu} = q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu,$$
$$T_4^{\mu\nu} = q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu,$$
$$T_5^{\mu\nu} = q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,$$

This is a basis of gauge-invariant tensors, but for $q_1 \cdot q_2 = 0$ it becomes degenerate: need one more structure: Tarrach (75)

$$T_6^{\mu\nu} = \left(q_1^2 q_3^\mu - q_1 \cdot q_3 q_1^\mu\right)\left(q_2^2 q_3^\nu - q_2 \cdot q_3 q_2^\nu\right)$$

# Inverse-amplitude method's input