

# Rare Kaon Decays $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ with 3 Flavours

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Lattice 2017

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# RBC & UKQCD Collaborations

## BNL and RBRC

Mattia Bruno  
Tomomi Ishikawa  
Taku Izubuchi  
Luchang Jin  
Chulwoo Jung  
Christoph Lehner  
Meifeng Lin  
Hiroshi Ohki  
Shigemi Ohta (KEK)  
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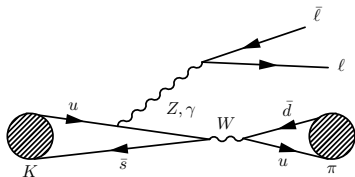
- “Prospects for a lattice computation of rare kaon decay amplitudes: I  $K \rightarrow \pi \ell^+ \ell^-$  decays”, N. H. Christ, X. Feng., A. Portelli and C. T. Sachrajda. arXiv:hep-lat\1507.03094
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- “Exploratory Lattice QCD Study of the Rare Kaon Decay  $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ ” Z. Bai, N. H. Christ, X. Feng., A. Lawson, A. Portelli and C. T. Sachrajda. arXiv:hep-lat\1701.02858.

# Outline

- 1 Introduction
  - Motivation
- 2 Lattice Methodology
  - Operators
  - Wick Contractions
  - Lattice Simulation
- 3 Renormalisation
  - Divergences
  - Matching

# Rare Kaon Decays

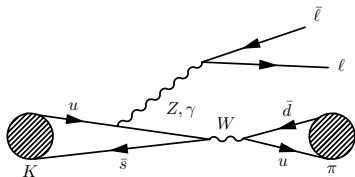
- The processes  $K \rightarrow \pi \ell \bar{\ell}$  proceed via a flavour changing neutral current (FCNC).
  - $s \rightarrow d \ell \bar{\ell}$  transition.
- FCNCs are forbidden at tree level in the SM.
- Ideal probes for new physics!



# Rare Kaon Decays

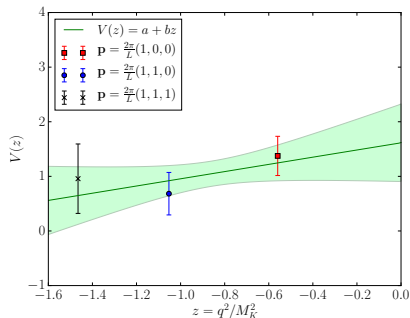
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- FCNCs are forbidden at tree level in the SM.
- Ideal probes for new physics!
- Six different decays with very different phenomenology:
 

<ul style="list-style-type: none"> <li>• <math>K_L \rightarrow \pi^0 \nu \bar{\nu}</math></li> <li>• <math>K_S \rightarrow \pi^0 \nu \bar{\nu}</math></li> <li>• <math>K^+ \rightarrow \pi^+ \nu \bar{\nu}</math></li> </ul>	<ul style="list-style-type: none"> <li>• <math>K_L \rightarrow \pi^0 \ell^+ \ell^-</math></li> <li>• <math>K_S \rightarrow \pi^0 \ell^+ \ell^-</math></li> <li>• <math>K^+ \rightarrow \pi^+ \ell^+ \ell^-</math></li> </ul>
--	--
- $K^+ \rightarrow \pi^+ \ell^+ \ell^-$  and  $K_S \rightarrow \pi^0 \ell^+ \ell^-$  long distance dominated.
  - Ideal target for a lattice calculation!



# Exploratory lattice results

- Studied  $K \rightarrow \pi \ell^+ \ell^-$  on coarse lattice ( $a^{-1} = 1.78$  GeV) with  $M_\pi \simeq 430$  MeV,  $M_K \simeq 625$  MeV, with a charm quark of mass  $m_c \simeq 530$  MeV.
- Obtained approximate  $q^2$  dependence of form factor,  $V(z)$ .



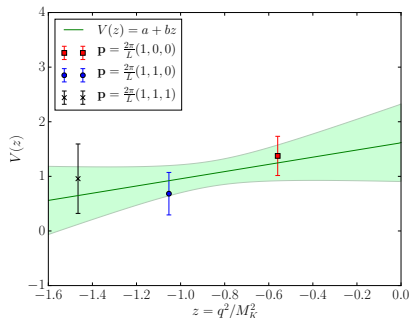
- Now wish to try physical calculation - computationally much more expensive!
  - Physical charm requires a fine lattice to avoid discretisation effects - integrate out?

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# Lattice Outlook

- Four flavour simulation:
  - Requires fine lattice to simulate physical charm (expensive!)
  - Lattice must be large enough to allow pion to propagate
    - $64^3 \times 128$  lattice available with  $a^{-1} = 2.36$  GeV
- Three flavour simulation:
  - Renormalisation more complicated!
  - Perturbation theory may not be completely reliable around  $\mu \sim m_c$ .
  - Computationally much cheaper for “physical” calculation.
    - $48^3 \times 96$  lattice with  $a^{-1} = 1.78$  GeV
- Aim to generate new data using new simulation framework: Grid.

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# Lattice Methodology

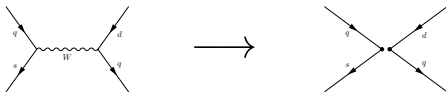
# $K \rightarrow \pi \ell^+ \ell^-$ Amplitude

- We compute the amplitude of  $K \rightarrow \pi \gamma^*$ ,

$$A_\mu(q^2) = \int d^4x \langle \pi(p) | T [J^\mu(0) H_W(x)] | K(k) \rangle + \langle \pi(p) | C_{7V} Q_{7V}^\mu | K(k) \rangle$$

- The effective Weak Hamiltonian for a  $|\Delta S| = 1$  transition is:

$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left( \sum_{i=1}^2 C_i (Q_i^u - Q_i^c) + \sum_{j=3}^{10} C_j Q_j + \mathcal{O} \left( \frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \right) \right).$$



[2] G. Isidori et al., Phys. Lett. B633 (2005) 75-83. arXiv:hep-lat/0506026

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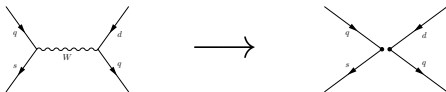
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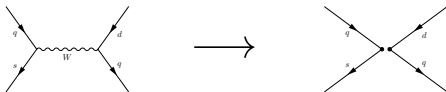
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- Also a small contribution from local operator

$$Q_{7V}^\mu = (\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) \bar{s} \gamma_\nu (1 - \gamma_5) d$$

- Negligible contribution from magnetic operator

$$Q_m^\mu = i \partial_\nu \bar{s} \sigma^{\nu\mu} (m_s (1 - \gamma_5) + m_d (1 + \gamma_5)) d$$

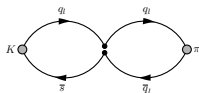
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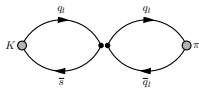
# Wick Contractions

- Performing Wick contractions with just  $H_W$ , we obtain 4 different classes of diagrams:

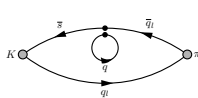
$$\langle \pi | H_W | K \rangle$$



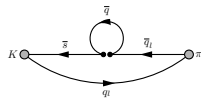
C



W



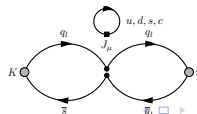
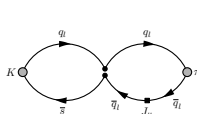
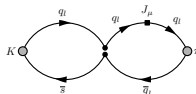
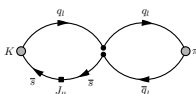
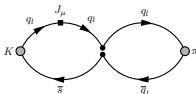
E



S

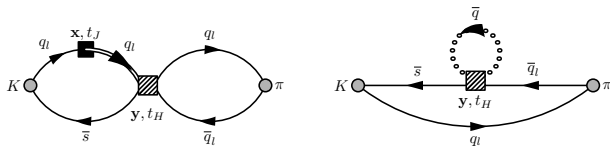
- Adding in the current, we obtain 5 diagrams per  $H_W$  class:

$$\int d^4x \langle \pi(p) | T [J^\mu(0) H_W(x)] | K(k) \rangle$$



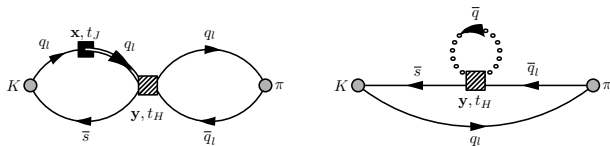


# Simulation Cost



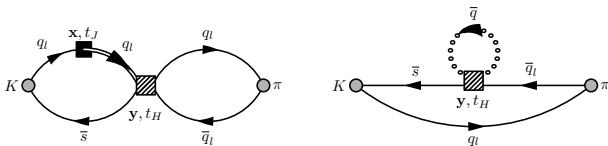
- Current: Sequential insertion (fixes  $t_J$ ).
- Loops: stochastically approximate diagonal of  $D^{-1}(x, y)$ .
  - Use low mode deflation.
  - Stochastic approximation necessary for high modes.
- Exploratory studies, used 128 configurations on  $24^3 \times 64$  lattice,  $a^{-1} = 1.78$  GeV:
  - 254 light inversions / config
  - 36 strange inversions / config
  - 182 charm inversions / config
- Prospects for physical point calculation very expensive!

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# Renormalisation

# Operator Renormalisation

$$A_\mu(q^2) = \int d^4x \langle \pi(p) | T [J_\mu(0) H_W(x)] | K(k) \rangle + \langle \pi(p) | C_{7V} Q_{7V}^\mu | K(k) \rangle$$

- If we choose  $J_\mu$  to be conserved - no renormalisation is required.
- For  $H_W$  we renormalise the bare operators  $Q_i$  non-perturbatively, e.g. using RI-SMOM.
- The Wilson coefficients  $C_i$  known at NLO in  $\overline{\text{MS}}$  scheme; we can use continuum perturbation theory to match to RI-SMOM.

$$Q_i \xrightarrow{\text{NPR}} Q_i^{\text{RI-SMOM}} \xrightarrow{\text{PT matching}} Q_i^{\overline{\text{MS}}}$$

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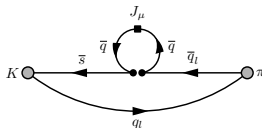
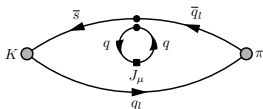
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# Divergences

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- Consider the Wick contractions:

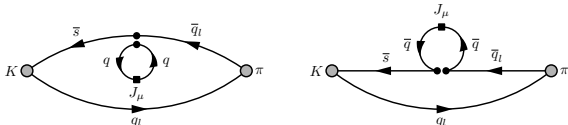


- These diagrams naively appear quadratically divergent!
- With conserved current electromagnetic gauge invariance implies divergence is logarithmic.
  - Cancelled by GIM mechanism.
- How to handle 3 flavour case?



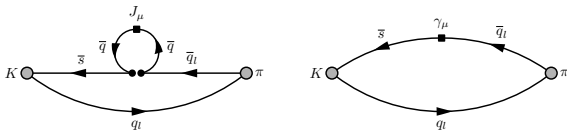
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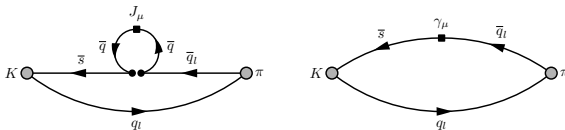


- With 3 flavours, charm contribution moved to local piece.
  - Must match the divergence from the lattice to the  $\overline{\text{MS}}$  scheme.
- Divergence in bilocal matrix element regulated by local operator:

$$\begin{aligned}
 \langle \pi | T [Q_i J^\mu] | K \rangle^{\overline{\text{MS}}}(\mu) &= Z_{ij}^{\text{RI} \rightarrow \overline{\text{MS}}} \left( Z_{jk}^{\text{RI}}(\mu_{\text{RI}}, a) \langle \pi | T [Q_k J^\mu] | K \rangle^{\text{lat}} \right. \\
 &\quad \left. - X_j(\mu_{\text{RI}}, a) \langle \pi | Q_{7V}^\mu | K \rangle^{\text{RI}} \right) \\
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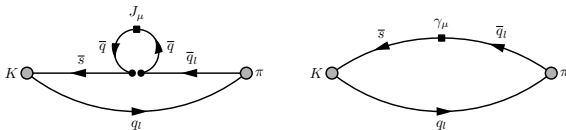


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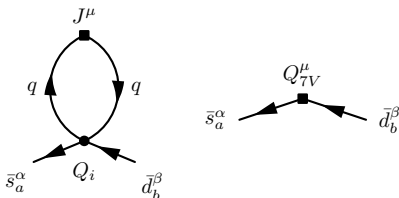


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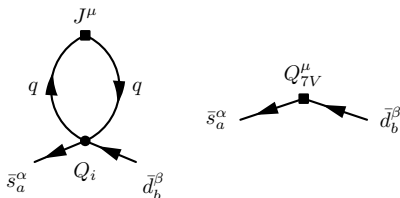
# 3 Flavour Renormalisation



- Step 1: convert divergence from lattice to RI-SMOM scheme:

$$\begin{aligned}
 \langle d | T [Q_i J^\mu] | s \rangle^{\text{RI}} \Big|_{p_i^2 = \mu_{\text{RI}}^2} &= Z_{ij}^{\text{RI}}(\mu_{\text{RI}}, a) \langle d | T [Q_j J^\mu] | s \rangle^{\text{lat}} \Big|_{p_i^2 = \mu_{\text{RI}}^2} \\
 &\quad - X_i(\mu_{\text{RI}}, a) \langle d | Q_{7V}^\mu | s \rangle^{\text{RI}} \Big|_{p_i^2 = \mu_{\text{RI}}^2} \\
 &= 0.
 \end{aligned}$$

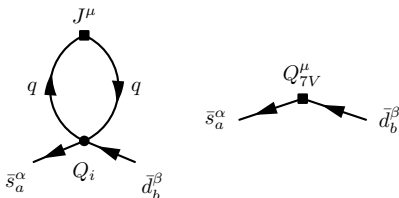
# 3 Flavour Renormalisation



- Step 2: Perturbative calculation to calculate  $Y_i$  to match to  $\overline{\text{MS}}$  scheme.

$$\langle d | T [Q_i J^\mu] | s \rangle^{\overline{\text{MS}}} \Big|_{p_i^2 = \mu_{\text{RI}}^2} = Y_i(\mu_{\text{RI}}, \mu) \langle d | Q_{7V}^\mu | s \rangle^{\text{RI}} \Big|_{p_i^2 = \mu_{\text{RI}}^2}.$$

# 3 Flavour Renormalisation



- Step 3: Divergences cancel between bilocal matrix element and Wilson coefficient of local operator

$$\mathcal{A}(K^+ \rightarrow \pi^+ \ell^+ \ell^-) = \sum_i C_i(\mu) \langle \pi | T [Q_i J^\mu] | K \rangle^{\overline{\text{MS}}}(\mu) + C_{7V}(\mu) \langle \pi | Q_{7V}^\mu | K \rangle.$$

[6] Buchalla et al., Rev. Mod. Phys. 68 (1996) 1125. arXiv:hep-ph/9512380.

# Summary

- We have developed theoretical techniques for calculating the long distance contributions to  $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ .
- Our exploratory calculations show that we can successfully extract the matrix elements in practice.
- Physical point simulation very expensive
  - 3-flavour calculation good intermediate step before 4-flavour physical lattice calculation is feasible.
- Development of Grid ongoing
  - Implementation of  $K \rightarrow \pi \ell^+ \ell^-$  contractions with Grid complete.
  - NPR code in progress.



Thank you!