Kramers-Wannier duality with Abelian Color Fluxes for the SU(2) principal chiral

christof.gattringer@uni-graz.at

Carlotta Marchis, Daniel Göschl

The idea of dual variables

- In several theories one can solve the complex action problem at finite μ by exactly mapping the partition sum to dual variables
- Dual variables are generalized world-lines for matter fields and generalized world-sheets for gauge degrees of freedom
- Often several different dual representations are possible[1]
- Symmetries of the underlying theory are converted into constraints for the dual variables
- Conserved charges become manifest as winding numbers of the dual variables
- Non-abelian d.o.f.s are a challenge which we here address with the new concept of Abelian Color Fluxes (ACF) for the example of the SU(2) principal chiral model[2]
  - How are non-abelian symmetries manifest in the dual representation?
  - How are charges represented?
  - What is the structure after a full Kramers-Wannier duality transformation?

[1] For a different dualization of the SU(2) PCR see: Rindlisbacher & de Forcrand, PoS LATTICE2015
[2] ACF can be generalized to gauge theories: NPB 913 (2017) and talk by Carlotta Marchis

SU(2) principal chiral model

- Partition sum:

$$Z \;=\; \int D[U]\, e^{-S} \quad , \qquad \int D[U] = \prod_x \int_{\text{SU(2)}} dU_x$$

- Action with two charges coupled to chemical potentials μ_1 and μ_2

$$S \;=\; -\frac{J}{2} \sum_{x,\nu} \text{Tr}\Big[e^{\delta_{\nu,d}\,\sigma^3\,[\,\mu_1 + \mu_2\,]/2}\, U_x\, e^{\delta_{\nu,d}\,\sigma^3\,[\,\mu_1 - \mu_2\,]/2}\, U^\dagger_{x+\hat{\nu}} + e^{-\delta_{\nu,d}\,\sigma^3\,[\,\mu_1 - \mu_2\,]/2}\, U^\dagger_x\, e^{-\delta_{\nu,d}\,\sigma^3\,[\,\mu_1 + \mu_2\,]/2}\, U_{x+\hat{\nu}} \Big]$$

- Using $U_x^{11} = U_x^{22\,\star}$ and $U_x^{12} = -U_x^{21\,\star}$ we find

$$S \;=\; -J \sum_{x,\nu} \sum_{a,b=1}^{2} M_\nu^{ab}\, U_x^{ab}\, U_{x+\hat{\nu}}^{ab\,\star}$$

with

$$M_\nu^{11} = e^{\delta_{\nu,d}\,\mu_1} \quad , \quad M_\nu^{12} = e^{\delta_{\nu,d}\,\mu_2} \quad , \quad M_\nu^{21} = e^{-\delta_{\nu,d}\,\mu_2} \quad , \quad M_\nu^{22} = e^{-\delta_{\nu,d}\,\mu_1}$$

Abelian Color Fluxes (ACF)

- We expand Z in $U_x^{ab}\, U_{x+\hat{\nu}}^{ab\star}$, referred to as "Abelian Color Fluxes"

$$
\begin{aligned}
Z &= \int D[U]\; e^{\,J \sum_{x,\nu} \sum_{a,b} M_\nu^{ab}\, U_x^{ab}\, U_{x+\hat{\nu}}^{ab\star}} = \int D[U] \prod_{x,\nu} \prod_{a,b} e^{\,J\, M_\nu^{ab}\, U_x^{ab}\, U_{x+\hat{\nu}}^{ab\star}} \\
&= \int D[U] \prod_{x,\nu} \prod_{a,b} \sum_{l_{x,\nu}^{ab}=0}^{\infty} \frac{(J\, M_\nu^{ab})^{l_{x,\nu}^{ab}}}{l_{x,\nu}^{ab}\,!} \left(U_x^{ab}\, U_{x+\hat{\nu}}^{ab\star} \right)^{l_{x,\nu}^{ab}} \\
&= \sum_{\{l\}} \prod_{x,\nu} \prod_{a,b} \frac{(J\, M_\nu^{ab})^{l_{x,\nu}^{ab}}}{l_{x,\nu}^{ab}\,!} \prod_x \int_{\mathrm{SU(2)}} dU_x \prod_{a,b} \left(U_x^{ab} \right)^{\sum_\nu l_{x,\nu}^{ab}} \left(U_x^{ab\star} \right)^{\sum_\nu l_{x-\hat{\nu},\nu}^{ab}}
\end{aligned}
$$

- The SU(2) matrices are integrated out with some explicit representation

$$
U_x = \begin{pmatrix} \cos\theta_x\, e^{i\alpha_x} & \sin\theta_x\, e^{i\beta_x} \\ -\sin\theta_x\, e^{-i\beta_x} & \cos\theta_x\, e^{-i\alpha_x} \end{pmatrix} \quad , \qquad \alpha_x, \beta_x \in [-\pi, \pi],\;\; \theta_x \in [0, \pi/2]
$$

- Constraints and additional weights

$\int d\alpha_x$, $\int d\beta_x$ $\Rightarrow$ 2 constraints per site , $\int d\theta_x$ $\Rightarrow$ site weights W_H

Worldline representation

- Partition sum

$$Z = \sum_{\{l\}} e^{\,\mu_1 \sum_x \left[l^{11}_{x,d} - l^{22}_{x,d} \right]} \, e^{\,\mu_2 \sum_x \left[l^{12}_{x,d} - l^{21}_{x,d} \right]} \, W_B[\,l\,] \, W_H[\,l\,]$$

$$\prod_x \delta\left(\sum_\nu [(\, l^{11}_{x,\nu} - l^{22}_{x,\nu}) - (\, l^{11}_{x-\hat{\nu},\nu} - l^{22}_{x-\hat{\nu},\nu})] \right) \delta\left(\sum_\nu [(\, l^{12}_{x,\nu} - l^{21}_{x,\nu}) - (l^{12}_{x-\hat{\nu},\nu} - l^{21}_{x-\hat{\nu},\nu})] \right)$$

- Introduce new dual variables $k^{\,1}_{x,\nu}, k^{\,2}_{x,\nu} \in \mathbb{Z}$ and $h^{\,1}_{x,\nu}, h^{\,2}_{x,\nu} \in \mathbb{N}_0$

$$l^{\,11}_{x,\nu} - l^{\,22}_{x,\nu} = k^{\,1}_{x,\nu} \qquad\qquad j^{\,12}_{x,\nu} - j^{\,21}_{x,\nu} = k^{\,2}_{x,\nu}$$

$$l^{\,11}_{x,\nu} + l^{\,22}_{x,\nu} = |\, k^{\,1}_{x,\nu} \,| + 2\, h^{\,1}_{x,\nu} \qquad\qquad j^{\,12}_{x,\nu} + j^{\,21}_{x,\nu} = |\, k^{\,2}_{x,\nu} \,| + 2\, h^{\,2}_{x,\nu}$$

- Final worldline form

$$Z = \sum_{\{k,h\}} e^{\,\mu_1 \sum_x k^{\,1}_{x,d}} \, e^{\,\mu_2 \sum_x k^{\,2}_{x,d}} \, W_B[\,k,h\,] \, W_H[\,k,h\,] \, \prod_x \delta\big(\nabla k^{\,1}_x\big) \; \delta\big(\nabla k^{\,2}_x\big)$$

$$\nabla k^{\,\alpha}_x \equiv \sum_\nu [\, k^{\,\alpha}_{x,\nu} - k^{\,\alpha}_{x-\hat{\nu},\nu} \,]$$

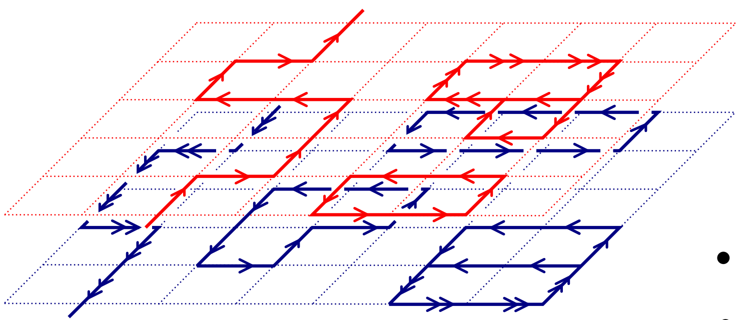
Geometrical interpretation

- Final form of worldline partition sum

$$Z = \sum_{\{k,h\}} e^{\,\mu_1 N_T \omega[k^1]} \, e^{\,\mu_2 N_T \omega[k^2]} \, W_B[\,k,h\,] \, W_H[\,k,h\,] \prod_x \delta\big(\nabla k_x^{\,1}\big) \; \delta\big(\nabla k_x^{\,2}\big)$$

$$W_B[k,h] = \prod_{x,\nu} \prod_{\alpha=1}^{2} \frac{J^{\,D_{x,\nu}^{\alpha}}}{(D_{x,\nu}^{\,\alpha} - h_{x,\nu}^{\,\alpha})! \, h_{x,\nu}^{\,\alpha}!} \qquad \omega[k^\alpha] = \text{temporal winding}$$

$$W_H[k,h] = \prod_x \frac{\prod_\alpha \left(\frac{1}{2} \sum_\nu [D_{x,\nu}^{\,\alpha} - D_{x-\hat{\nu},\nu}^{\,\alpha}]\right)!}{\left(1 + \sum_\alpha \sum_\nu [D_{x,\nu}^{\,\alpha} - D_{x-\hat{\nu},\nu}^{\,\alpha}]\right)!} \qquad D_{x,\nu}^{\,\alpha} \equiv |\,k_{x,\nu}^{\,\alpha}\,| + 2\,h_{x,\nu}^{\,\alpha}$$

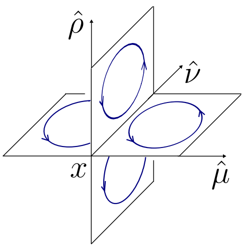


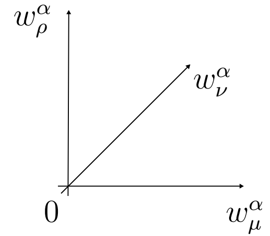
- Theory of 2 conserved fluxes $k^{\,1}, k^{\,2}$
- Weight W_H and auxiliary variables $h^{\,1}, h^{\,2}$ implement SU(2)

Steps for a full dualization (Kramers - Wannier)

- Resolve the contraints by generating the conserved fluxes $k_{x,\nu}^{\alpha}$ with new, unconstrained variables:
  - Fluxes $n_{x,\mu\nu}^{\alpha} \in \mathbb{Z}$ around plaquettes
  - Global defect fluxes w_{ν}^{α} for the winding

$$k_{x,\nu}^{\alpha} \; = \; \sum_{\rho:\,\nu<\rho} \left[n_{x,\nu\rho}^{\alpha} - n_{x-\hat{\nu},\nu\rho}^{\alpha} \right] - \sum_{\mu:\,\mu<\nu} \left[n_{x,\mu\nu}^{\alpha} - n_{x-\hat{\mu},\mu\nu}^{\alpha} \right] \; + \; \theta(x,\nu)\, w_{\nu}^{\alpha}$$





- Identify counterparts on the dual lattice for the new variables $n_{x,\mu\nu}^{\alpha}$ and w_{ν}^{α}, as well as for the auxiliary variables $h_{x,\mu\nu}^{\alpha}$.

Dualization in 3 dimensions (other dimensions equivalent)

- Variables on the dual lattice

$$n^{\alpha}_{x,\mu\nu} \longrightarrow l^{\alpha}_{x,\mu} \in \mathbb{Z}$$ dynamical variables on dual links

$$h^{\alpha}_{x,\mu} \longrightarrow a^{\alpha}_{x,\mu\nu} \in \mathbb{N}_0$$ auxiliary variables on dual plaquettes

$$w^{\alpha}_{\mu} \longrightarrow b^{\alpha}_{\mu\nu} \in \mathbb{Z}$$ boundary condition on dual plaquettes

$$P^{\alpha}_{x,\mu\nu} \equiv | \, l^{\alpha}_{x,\mu} - l^{\alpha}_{x+\hat{\mu},\nu} - l^{\alpha}_{x+\hat{\nu},\mu} + l^{\alpha}_{x,\nu} \; + \; \theta(x,\mu,\nu) \, b^{\alpha}_{\mu\nu} \, | \; + 2 \, a^{\alpha}_{x,\mu\nu}$$

- Fully dualized partition sum in 3D

$$Z \;=\; \sum_{\{b\}} e^{\,\mu_1 N_T \, b^1_{12}} \; e^{\,\mu_2 N_T \, b^2_{12}} \sum_{\{l,a\}} M_B[l,a,b] \, M_H[l,a,b]$$

$$M_B[l,a,b] \;=\; \prod_{\text{plaquettes } p} \; \prod_{\alpha=1}^{2} \frac{J^{\,P^{\alpha}_p}}{(P^{\alpha}_p - a^{\alpha}_p)! \;\; a^{\alpha}_p \, !}$$

$$M_H[l,a,b] \;=\; \prod_{\text{cubes } c} \frac{\prod_{\alpha} \left(\frac{1}{2} \sum_{p \in \partial c} P^{\alpha}_p \right)!}{\left(1 + \frac{1}{2} \sum_{\alpha} \sum_{p \in \partial c} P^{\alpha}_p \right)!}$$

strong coupling ⇔ weak coupling

Summary

- We explore new representations of the SU(2) principal chiral model with chemical potentials coupled to two conserved charges
- Worldline representation
  - We expand the Boltzmann factor in Abelian Color Fluxes
  - All terms of the partition sum are real and positive: complex action problem solved
  - Theory of two worldlines interacting with each other via site weights $\Rightarrow$ SU(2)
- Full Kramers - Wannier dualization
  - Constraints can be resolved by introducing new plaquettes variables and defect lines
  - Dual theory of links variables, with interactions on plaquettes and on cubes for implementing SU(2)
  - Dualization in the full sense: strong coupling $\Leftrightarrow$ weak coupling
- Exploratory study of simulating the different forms has started