Development of Grassmann higher order tensor renormalization group

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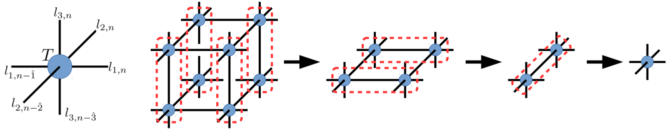
Introduction

- Tensor Renormalization Group (TRG) [M. Levin and C. P. Nave, Phys.Rev.Lett.99 (2007)]
  - Deterministic data compression
  - Completely free of the sign problem
- Outline of the TRG
  - Tensor network representation

$$Z = \sum_{\{l\}} \prod_n T_{l_{1,n} l_{1,n-\hat{1}} l_{2,n} l_{2,n-\hat{2}} l_{3,n} l_{3,n-\hat{3}}}$$

  - "Block Spin transformation" for the tensor network

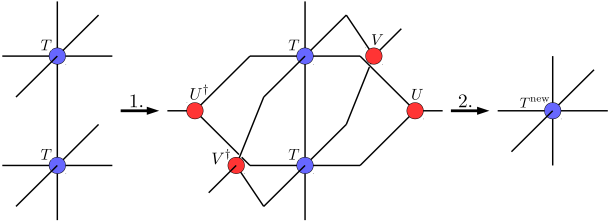


- TRG study of relativistic QFTs on the lattice
  - 2D ϕ^4 model [Y. Shimizu, Chin.J.Phys.50 (2012)]
  - 2D Schwinger model [Y. Shimizu and Y. Kuramashi, Phys.Rev.D90 (2014); Phys.Rev.D90 (2014)]
  - 2D $N_f = 1$ Gross–Neveu model [S. Takeda and Y. Yoshimura, Prog.Theor.Exp.Phys.2015]
  - 3D free Wilson fermion system [R. Sakai and S. Takeda, Lattice 2016]

    $\rightarrow$ expectation, correlation function, second renormalization

Higher order TRG [Xie et al. 2012]

- The procedure of HOTRG:
  1. Insert "isometries" U, V into the network.
  2. Contracting T, U, and V, obtain a new tensor T^{new}.



- If the bond dimension is D, the isometries can be considered $D^2 \times D_{\text{cut}}$ matrix $(D_{\text{cut}} \leq D^2)$.
- D_{cut} is the keeping dimension for the truncation, and determines the accuarcy of this approximation.
- If $D_{\text{cut}} = D^2$,

$$UU^\dagger = VV^\dagger = \text{identity}$$

- Computation time $\propto D_{\text{cut}}^{11}$, memory requirement $\propto D_{\text{cut}}^{6}$

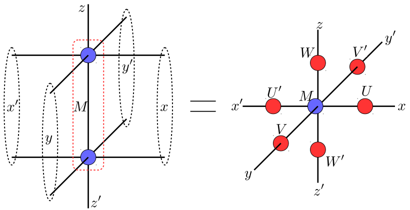
Higher order SVD

- Isometries are made via higher order singular value decomposition.

$$M_{xx'yy'zz'} = \sum_{ijklmn} S_{ijklmn} U_{xi} U'_{x'j} V_{yk} V'_{y'l} W_{zm} W'_{z'n}$$

U, V, W, U', V', W' are unitary, and S possesses the orthogonality

$$\sum_{jklmn} S_{ijklmn} S_{i'jklmn} \propto \delta_{ii'}$$



For example:

$$\sum_{x'yy'zz'} M^*_{x_1x'yy'zz'} M_{x_2x'yy'zz'} = \sum_i U^*_{x_1i} \lambda_i U_{x_2i}$$

3D free Wilson fermion

- Wilson fermion action (Wilson parameter: $r = 1$)

$$S = \sum_{n,n'} \bar{\psi}_n D^W_{n,n'} \psi_{n'}, \;\; D^W_{n,n'} = (m+3)\delta_{n,n'} - \frac{1}{2} \sum_{\mu,\pm} (1 \pm \gamma_\mu)\, \delta_{n,n' \pm \hat{\mu}}$$

- Gamma matrixes

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- Patition function

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \prod_n \left(\prod_s e^{-(m+3)\bar{\psi}_{n,s}\psi_{n,s}} \prod_\mu e^{\bar{\chi}_{\mu,n+\hat{\mu},1}\chi_{\mu,n,1}} e^{\bar{\chi}_{\mu,n,2}\chi_{\mu,n+\hat{\mu},2}} \right)$$

$$\chi_{1,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi_n, \chi_{2,n} = \psi_n, \chi_{3,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \psi_n$$

Tensor network representation

The procedure to transform into tensor network representation:

1. Expande each exponential binomially.
2. Insert new Grassmann numbers.
3. Separate original Grassmann numbers at other sites.
4. Integrate out original Grassmann numbers.

For example:

$$
\begin{aligned}
e^{\bar{\chi}_{\mu,n+\hat{\mu},1}\chi_{\mu,n,1}} &= \sum_{l_{\mu,n,1}=0}^{1} \left(\bar{\chi}_{\mu,n+\hat{\mu},1}\chi_{\mu,n,1}\right)^{l_{\mu,n,1}} \\
&= \sum_{l_{\mu,n,1}} \int \left(d\eta_{\mu,n,1}\eta_{\mu,n,1}\right)^{l_{\mu,n,1}} \left(\bar{\chi}_{\mu,n+\hat{\mu},1}\chi_{\mu,n,1}\right)^{l_{\mu,n,1}} \\
&= \sum_{l_{\mu,n,1}} \int \left(\chi_{\mu,n,1}d\eta_{\mu,n,1}\right)^{l_{\mu,n,1}} \left(\bar{\chi}_{\mu,n+\hat{\mu},1}\eta_{\mu,n,1}\right)^{l_{\mu,n,1}}
\end{aligned}
$$

- $\chi_{\mu,n,1}d\eta_{\mu,n,1}$ can be moved without -1 from the anticommutation.
- original $\chi_{\mu,n,1}$'s position is replaced by $\eta_{\mu,n,1}$.

Grassmann HOTRG

Corecting $\chi_{\mu,n,s} d\eta_{\mu,n,s}, \bar{\chi}_{\mu,n,s} d\bar{\eta}_{\mu,n,s}$, integrate out $\psi_n, \bar{\psi}_n$.

$$\mathcal{T}_{l_{1,n}l_{1,n-\hat{1}}l_{2,n}l_{2,n-\hat{2}}l_{3,n}l_{3,n-\hat{3}}} = T_{l_{1,n}l_{1,n-\hat{1}}l_{2,n}l_{2,n-\hat{2}}l_{3,n}l_{3,n-\hat{3}}} \cdot$$
$$\prod_{\mu} d\bar{\eta}_{\mu,n,2}^{l_{\mu,n,2}} d\eta_{\mu,n,1}^{l_{\mu,n,1}} d\eta_{\mu,n,2}^{l_{\mu,n-\hat{\mu},2}} d\bar{\eta}_{\mu,n,1}^{l_{\mu,n-\hat{\mu},1}} \left(\bar{\eta}_{\mu,n+\hat{\mu},1}\eta_{\mu,n,1}\right)^{l_{\mu,n,1}} \left(\bar{\eta}_{\mu,n,2}\eta_{\mu,n+\hat{\mu},2}\right)^{l_{\mu,n,2}}$$

- $T_{l_{1,n}l_{1,n-\hat{1}}l_{2,n}l_{2,n-\hat{2}}l_{3,n}l_{3,n-\hat{3}}}$ satisfies

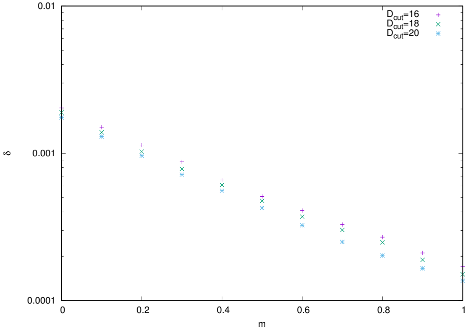
$$\sum_{\mu,s} (l_{\mu,n,1} + l_{\mu,n-\hat{\mu},s}) \bmod 2 \neq 0 \Rightarrow T_{l_{1,n}l_{1,n-\hat{1}}l_{2,n}l_{2,n-\hat{2}}l_{3,n}l_{3,n-\hat{3}}} = 0$$

- Simirarly, including out $\psi_n, \bar{\psi}_n$ for all sites,

$$Z = \sum_{\{l\}} \prod_n \mathcal{T}_{l_{1,n}l_{1,n-\hat{1}}l_{2,n}l_{2,n-\hat{2}}l_{3,n}l_{3,n-\hat{3}}}$$

- HOTRG procedure can be done like making tensor network representation.

Result: free energy



- The defference between our result and exact value

$$\delta = \frac{\ln Z_{\text{exact}} - \ln Z(D_{\text{cut}})}{\ln Z_{\text{exact}}}$$

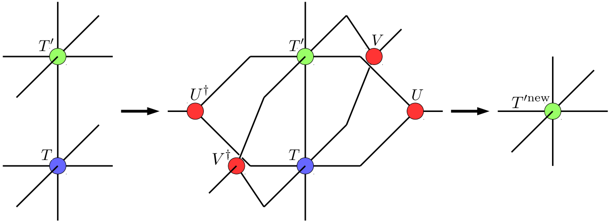
- $V = 64^3$
- The accuracy increase as D_{cut} is enlarged.

Expectation value

- Impure tensor representation for $\langle \bar{\psi}\psi \rangle$:

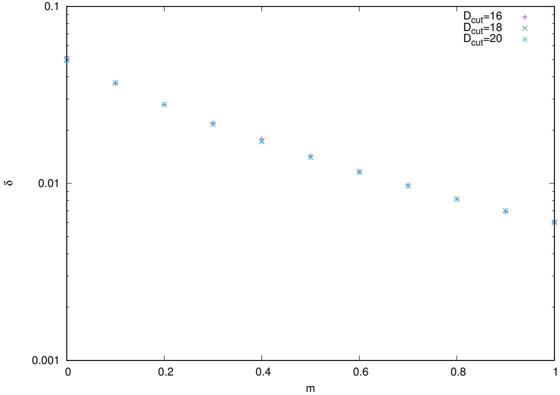
$$\langle \bar{\psi}_{n_0}\psi_{n_0} \rangle = \frac{1}{Z}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S}\bar{\psi}_{n_0}\psi_{n_0}$$
$$= \frac{1}{Z}\sum_{\{l\}} \mathcal{T}'_{n_0} \prod_{n\neq n_0} \mathcal{T}_n$$

- Impure tensor contraction



- Isometries U, V are made from T.

Result: chiral condensate



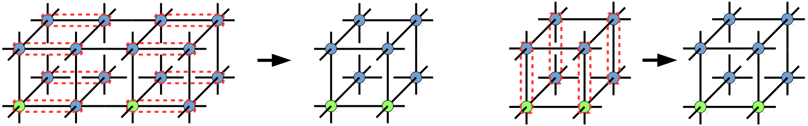
- $V = 64^3$
- The accuracy almost unchanges as D_{cut} is enlarged.

Correlation function

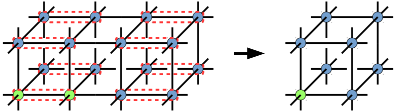
- Impure tensor representation for $\langle\bar{\psi}_{n_1,1}\psi_{n_2,2}\rangle$:

$$\langle\bar{\psi}_{n_1,1}\psi_{n_2,2}\rangle = \frac{1}{Z}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S}\bar{\psi}_{n_1,1}\psi_{n_2,2}$$
$$= \frac{1}{Z}\sum_{\{l\}}\mathcal{T}'_{n_1}\mathcal{T}''_{n_2}\prod_{n\neq n_1, n\neq n_2}\mathcal{T}_n$$

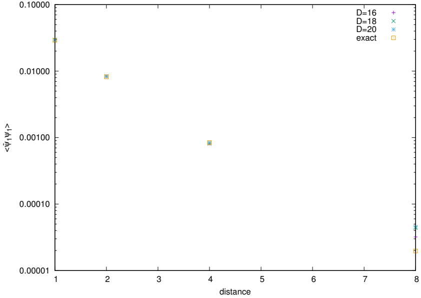
- Basically, each impure tensor are renormalized independently.

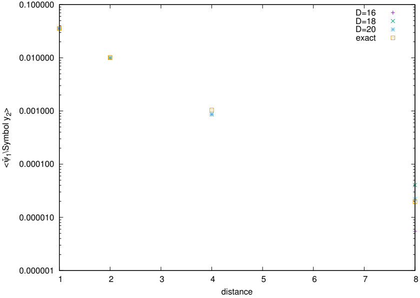


- Someday, two impure tensors are combined one tensor.



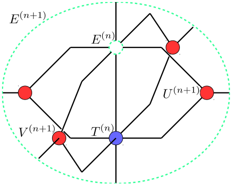
Result: correlation function





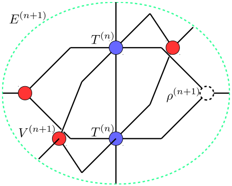
- $V = 64^3, m = 1$
- The accuracy does not increase as $D_{\rm cut}$ is enlarged.
- At long distance, the accuracy decrease.

- Second renormlization group (SRG) is the algorithm to take in global effects approximately.
- Environment tensor



$T^{(n)}$ is the tensor renormalized for n times.

- Density matrix

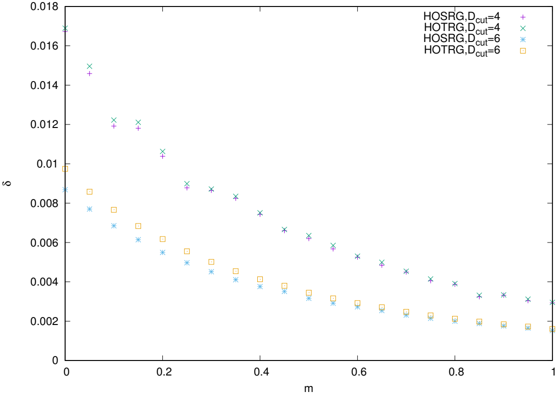


$$Z = \rho^{(n+1)} \cdot U^{(n+1)}$$

- singular value decomposition

$$\rho^{(n+1)} = u\sigma v^{\dagger} \Rightarrow U^{(n+1)} = vu^{\dagger}$$

Result: HOSRG



- $V = 64^3$
- The accuracy of HOSRG increase as compared with the accuracy of HOTRG.

Summary

- GHOTRG can caluculate the chairal condensate and fermion corelator in 3D free Wilson fermion as impure tensor formalization.
- By the GHOSRG, the Free Energy accuration increase.

In future→

- Applying GHOSRG scheme for tensor network including impure tensors to caluculate expectation values and correlation functions.
- Verifying that GHOTRG is availavle in more complicated 3D model