# $\eta$ and $\eta'$ masses and decay constants

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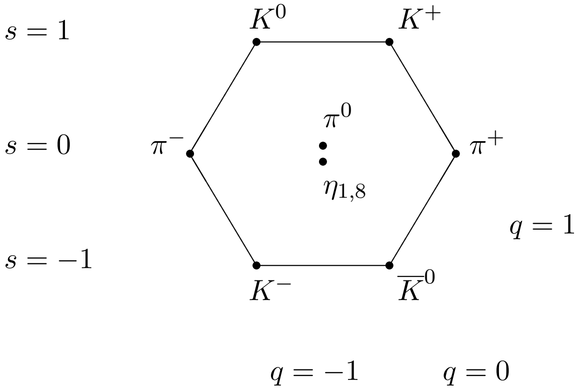
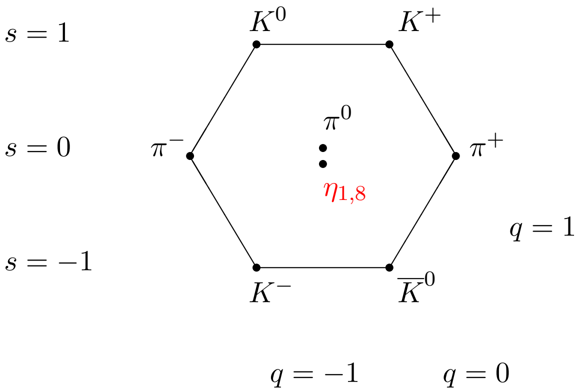
# Outline

# Quark model




$$SU(3) : 3 \otimes \bar{3} = 8 \oplus 1$$

# Quark model




$$SU(3) : 3 \otimes \bar{3} = 8 \oplus 1$$

# $\eta/\eta'$ mixing

- Simple $SU(3)$ flavour symmetry fails: Predicted singlet and octet particles

$$\eta_1 = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}, \qquad \eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}} \tag{1}$$

- In reality, $\eta$ and $\eta'$ are not flavour eigenstates:

$$U(\theta,\theta')\begin{pmatrix}\mathcal{O}_{88} & \mathcal{O}_{81}\\ \mathcal{O}_{18} & \mathcal{O}_{11}\end{pmatrix}U^T(\theta,\theta') = \begin{pmatrix}\mathcal{O}_{\eta\eta} & 0\\ 0 & \mathcal{O}_{\eta'\eta'}\end{pmatrix} \tag{2}$$

with mixing angles $\theta$ and $\theta'$,

$$\eta = \cos\theta\eta_8 - \sin\theta\eta_1, \qquad \eta' = \sin\theta'\eta_8 + \cos\theta'\eta_1 \tag{3}$$

- In a flavour symmetric world: $\eta = \eta_8 = \pi^0$

# The QCD vacuum: Axial symmetry breaking

- $\eta'$ becomes heavy compared to the octet mesons due to anomalous breaking of $U_A(1)$ axial symmetry.
- Witten and Veneziano could first explain the large mass of the $\eta'$ by relating it to the topological susceptibility $\chi$:

$$\frac{f_\pi^2 m_{\eta'}^2}{N_f} = \chi\Big|_{N_f=0} \tag{4}$$

(in the t'Hooft limit of $N_c \to \infty$)

- Sensitive to topology of the gluonic vacuum
  $\Rightarrow$ long autocorrelation times

# Disconnected loops

- When performing the Wick contractions for $\mathcal{O}_{88}$, $\mathcal{O}_{18}$, $\mathcal{O}_{81}$ and $\mathcal{O}_{11}$, disconnected correlators arise:

$$D_{f_1 f_2}(\delta t) = \frac{1}{N_t} \sum_t L_{f_1}(t) L_{f_2}(t + \delta t). \tag{5}$$

- Disconnected loops are "all-to-all": $L(t) = \operatorname{tr} \gamma_5 D^{-1}_{xx}$
  $\Rightarrow$ stochastic estimation
- Noise reduction:
  - time dilution: Put random sources at every 4th time slice
  - Hopping parameter expansion: use locality of the Wilson Dirac operator and expand in small $\kappa$
    $\Rightarrow$ using two and four applications for the pseudoscalar and axialvector loops, respectively

# The flavour symmetric case

- no mixing, trivial matrix diagonalization:

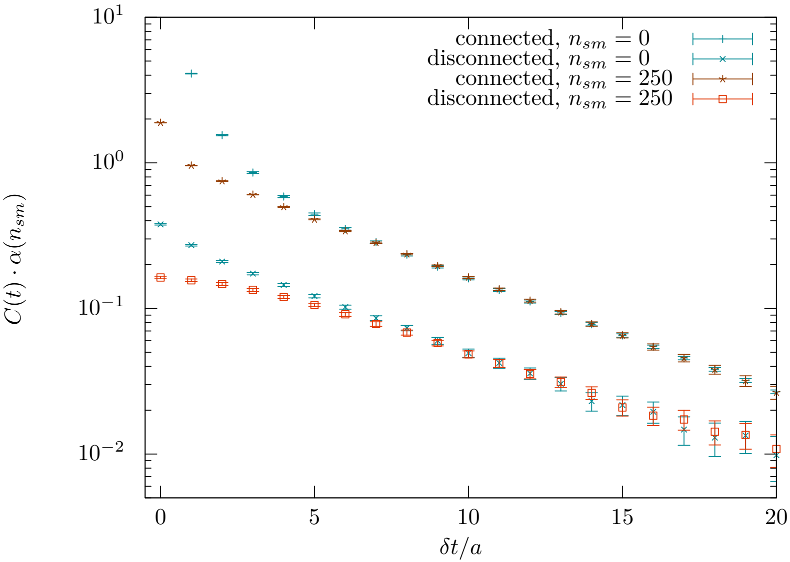
$$C_\eta(t) = C_\pi(t) = C_{con}(t), \qquad C_{\eta'}(t) = C_{con}(t) - N_f D(t) \tag{6}$$

- Disconnected correlator asymptotically goes with $\exp(-m_\pi t)$:

$$D(t) = \frac{1}{N_f}(C_{con}(t) - C_{\eta'}(t)) \tag{7}$$

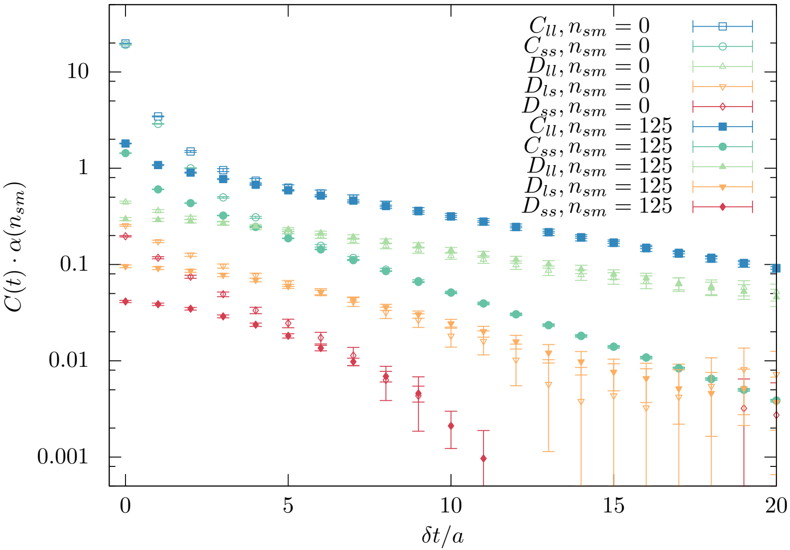
$\Rightarrow$ fitting the $\eta'$ amounts to fitting a double-exponential
$\Rightarrow$ small time behaviour encodes $\eta'$ physics!

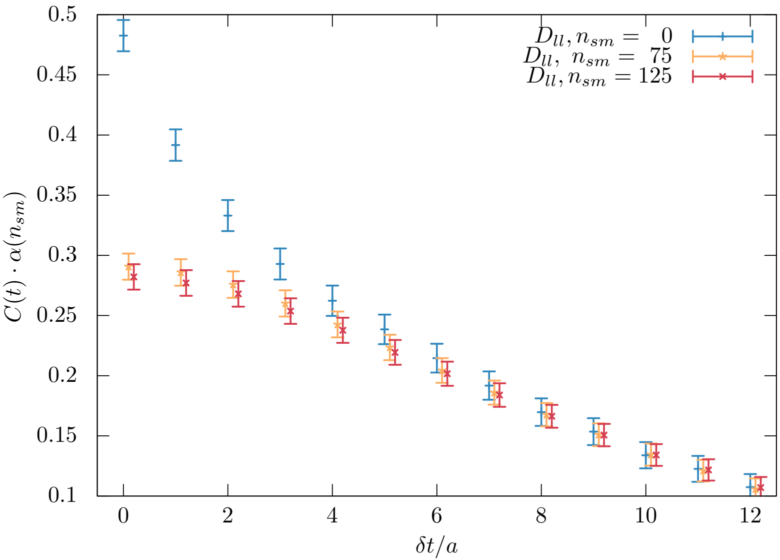
- excited states???




$\Rightarrow$ Controlling excited states is crucial for reliable analysis

# Excited states: $N_f = 2 + 1$




$D_{ll}, n_{sm} = 0$
$D_{ll}, n_{sm} = 75$
$D_{ll}, n_{sm} = 125$
$C(t) \cdot \alpha(n_{sm})$
$\delta t/a$
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
2
4
6
8
10
12

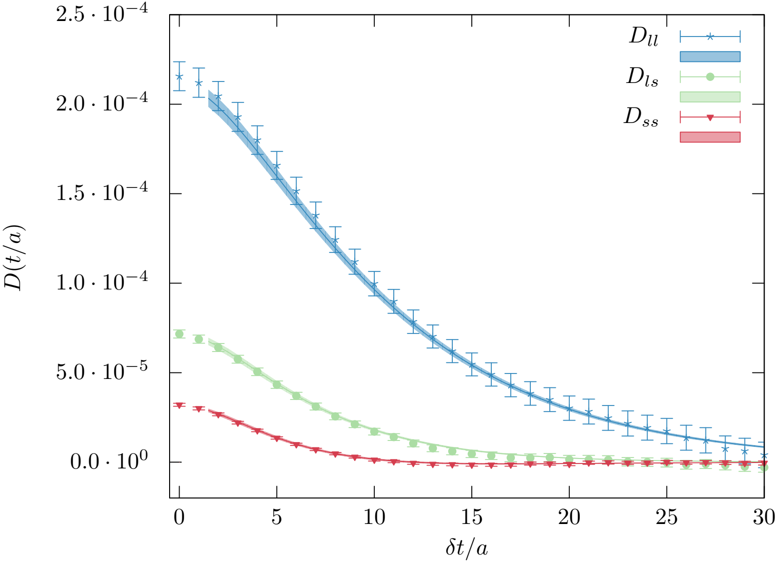
# $N_f = 2 + 1$: A fitting approach

- Usual analysis solves the GEVP

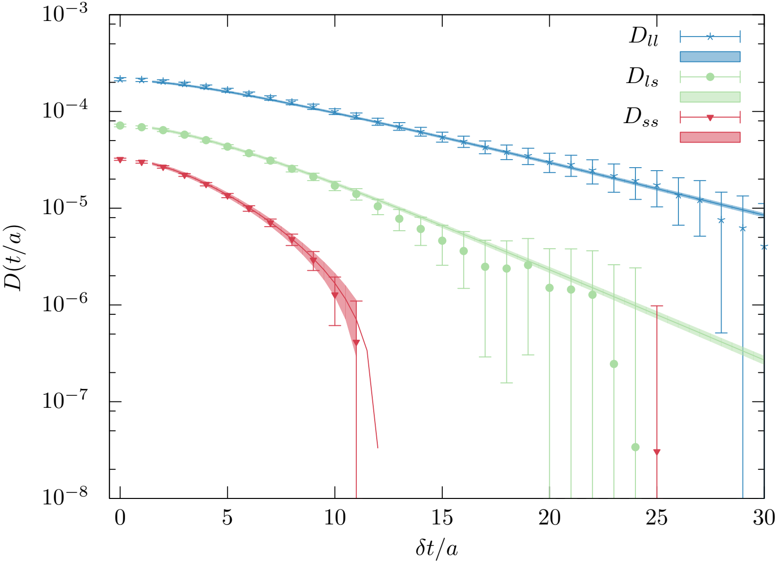
$$U(\theta, \theta') \begin{pmatrix} \mathcal{O}_{88} & \mathcal{O}_{81} \\ \mathcal{O}_{18} & \mathcal{O}_{11} \end{pmatrix} U^T(\theta, \theta') = \begin{pmatrix} \mathcal{O}_{\eta\eta} & 0 \\ 0 & \mathcal{O}_{\eta'\eta'} \end{pmatrix} . \qquad (8)$$

- Alternatively: Solve for the disconnected correlators $D_{\ell\bar{\ell}}$, $D_{\ell\bar{s}} = D_{s\bar{\ell}}$ and $D_{s\bar{s}}$
  $\Rightarrow$ obtain (complicated) fit forms, depending on ten parameters:
  $A_{\eta^{(\prime)}}$, $m_{\eta^{(\prime)}}$, $\theta^{(\prime)}$ and the connected correlators, $C_\pi(t)$ and $C_{s\bar{s}}(t)$
- Combined fit to all disconnected correlators
- Connected correlators usually determined much more precisely: can be fitted separately and reduces parameter space to six.

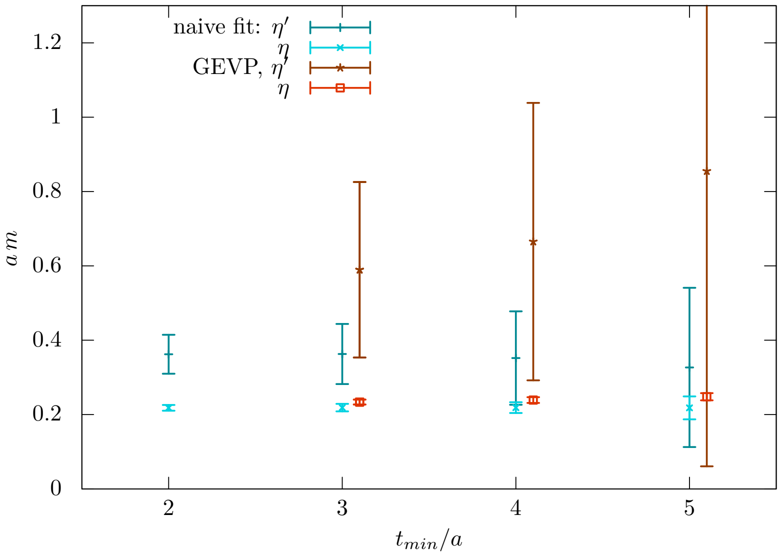
# Disconnected fits



# Disconnected fits



# first fits for the $\eta'$ masses



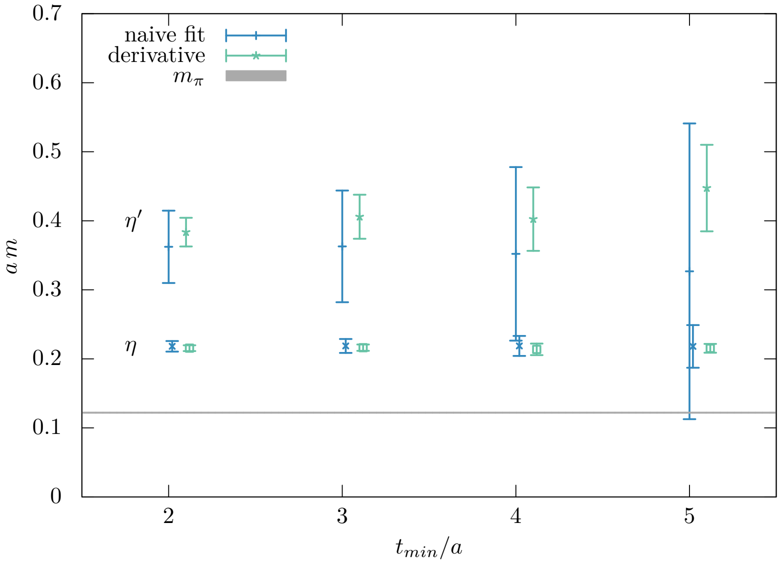
# Tricks for noise reduction during analysis

- Derivative trick: Fit to  hep-lat/0701005,arxiv:0909.3255

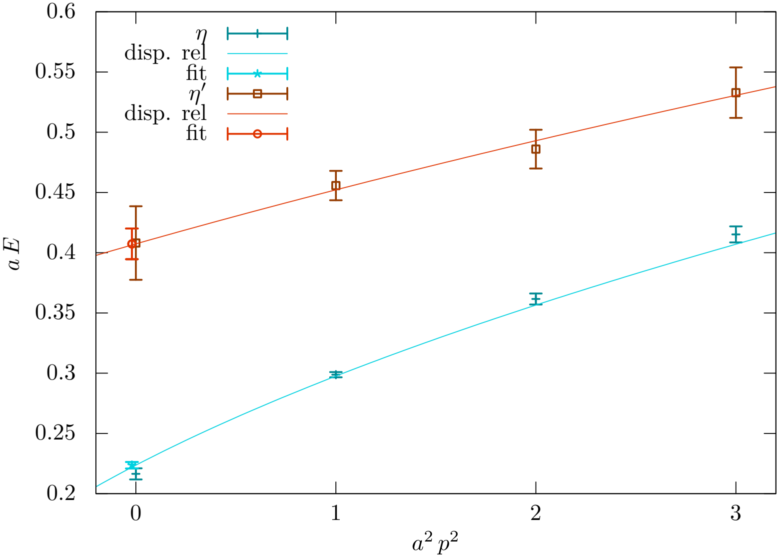
$$-a\hat{\partial}_4 C(\delta t) \approx a\left(C(\delta t) - C(\delta t + a)\right) \propto a\exp(-m\delta t) \quad (9)$$

  removes constant shifts (e.g., from topological objects) and correlations.

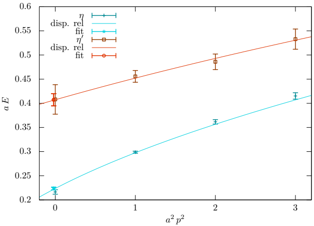
- for the masses: use dispersion relation to gain statistics


naive fit
derivative
$m_\pi$
$\eta'$
$\eta$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$a\,m$
2
3
4
5
$t_{min}/a$

- Started with two CLS $N_f = 2 + 1$ ensembles with non-perturbatively improved Wilson fermions Symanzik gauge action:
  - **U103** at the flavour symmetric point, $m_\pi = m_K = 415\,\mathrm{MeV}$: $\approx$ 1200 configurations, separated by 16 MDU, 48 stochastic estimates per config
  - **H105** $m_\pi = 282\,\mathrm{MeV}, m_K = 467\,\mathrm{MeV}$: $\approx$ 700 configurations, separated by 16 MDU, 96 stochastic estimates per config
- $\beta = 3.4$, $a \approx 0.086\,\mathrm{fm}$
- errors obtained via a binned jackknife analysis



0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
$a\,E$
0
1
2
3
$a^2\,p^2$
$\eta$
disp. rel
fit
$\eta'$
disp. rel
fit



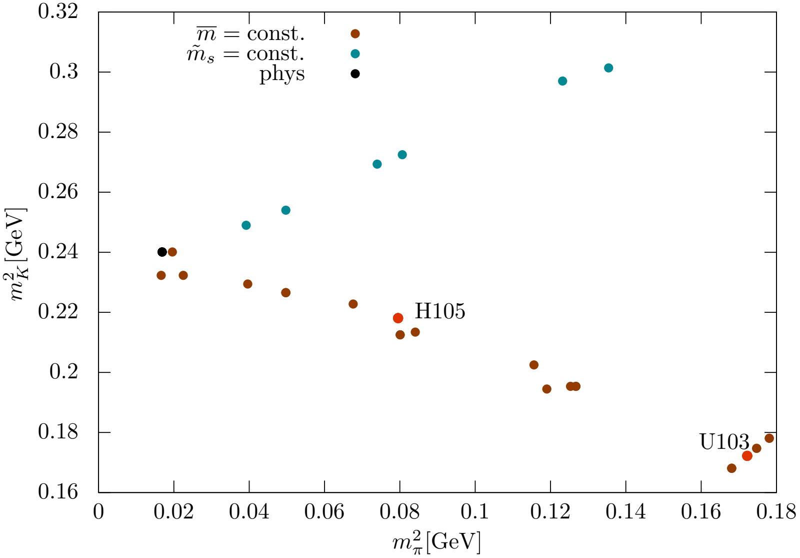

- **U103** (symmetric point)

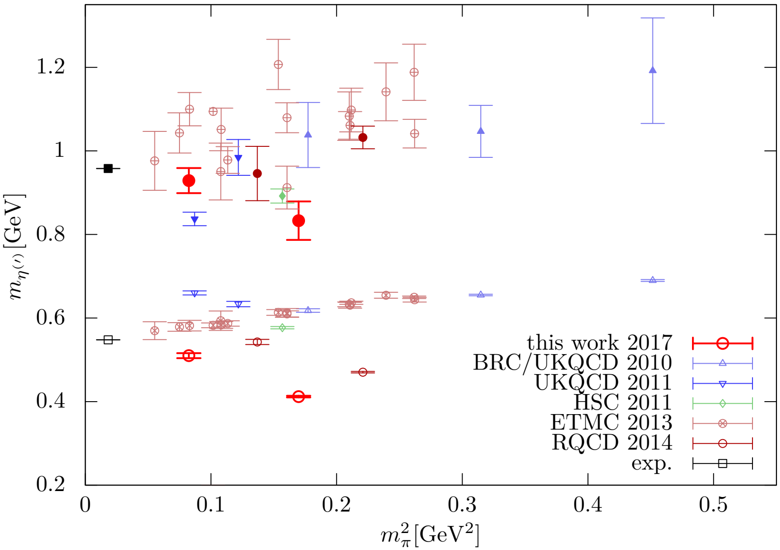
$$m_\eta = 412(2)\,\text{MeV} \qquad m_{\eta'} = 833(46)\,\text{MeV} \tag{10}$$

- **H105** $(m_\pi = 287\,\text{MeV},\ m_K = 487\,\text{MeV})$

$$m_\eta = 510(6)\,\text{MeV} \qquad m_{\eta'} = 929(30)\,\text{MeV} \tag{11}$$

Results preliminary and errors are purely statistical.


$\overline{m}$ = const.
$\tilde{m}_s$ = const.
phys
H105
U103
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
$m_K^2$ [GeV]
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
$m_\pi^2$ [GeV]



1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
$m_{\eta^{(\prime)}}$ [GeV]
$m_\pi^2$ [GeV$^2$]
this work 2017
BRC/UKQCD 2010
UKQCD 2011
HSC 2011
ETMC 2013
RQCD 2014
exp.

- Need pseudoscalar-axialvector matrix element to annihilate with the vacuum:

$$\langle 0 \mid \partial_\mu A_\mu^{1,8} \mid \eta^{(\prime)}(0)\rangle = f_{\eta^{(\prime)}}^{1,8} m_{\eta^{(\prime)}}^2 \tag{12}$$

- decay constants usually parameterized by $f_1$, $f_8$ and two angles $\theta_8$ and $\theta_1$

$$\begin{pmatrix} f_\eta^8 & f_\eta^1 \\ f_{\eta'}^8 & f_{\eta'}^1 \end{pmatrix} = \begin{pmatrix} f_8 \cos\theta_8 & -f_1 \sin\theta_1 \\ f_8 \sin\theta_8 & f_1 \cos\theta_1 \end{pmatrix} \tag{13}$$

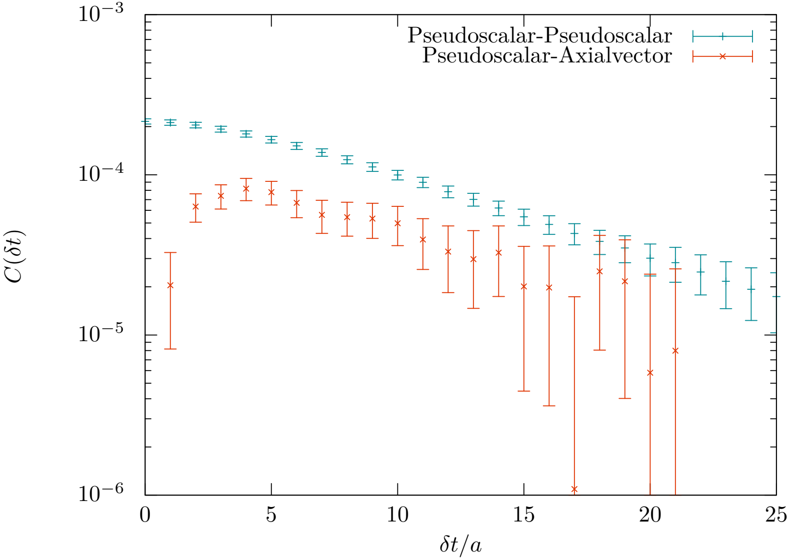
- singlet renormalization: difference known perturbatively to two loops: arxiv:1604.05827,arxiv:1610.06744

$$Z_A^1 = Z_A^{8,np} + g_0^4 z_A^{1,pt} + \mathcal{O}(g_0^6) \tag{14}$$

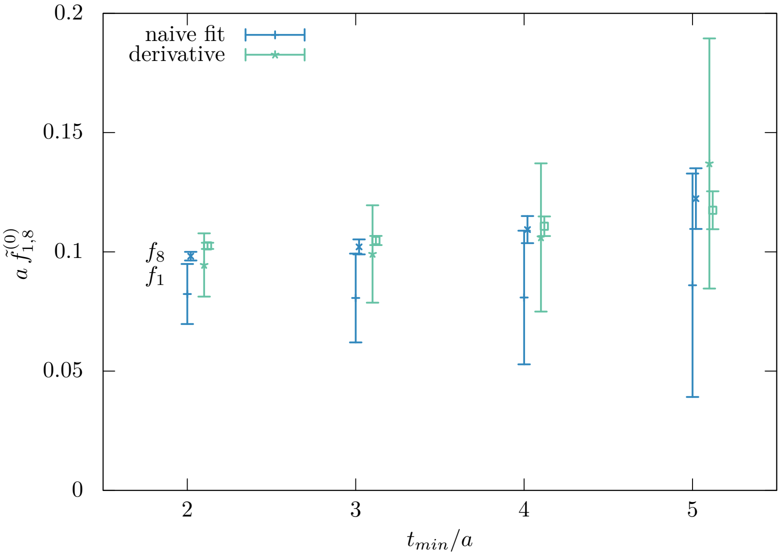
- $\mathcal{O}(a)$ improvement (neglecting small $\tilde{b}_A$ coefficient):

arxiv:1502.04999,arxiv:1607.07090

$$f_{1,8} = (1 + b_A a\overline{m}) Z_A^{1,8} (\tilde{f}_{1,8}^{(0)} + c_A a \tilde{f}_{1,8}^{(1)}) \tag{15}$$

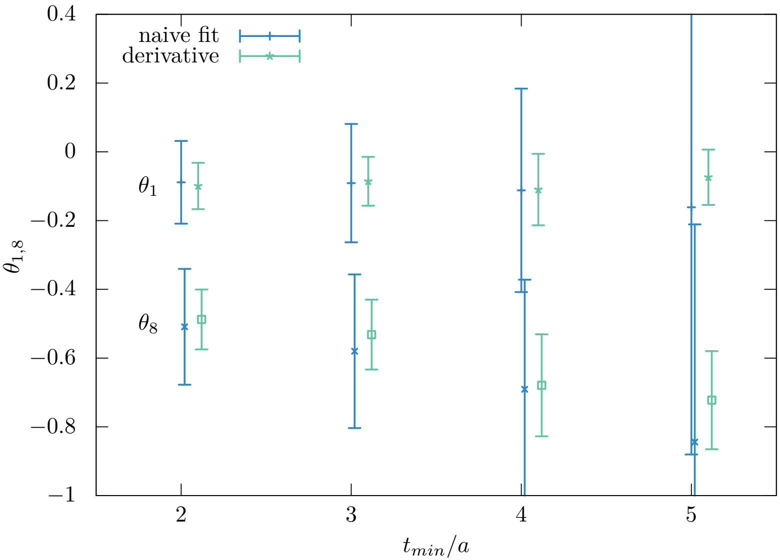


Pseudoscalar-Pseudoscalar
Pseudoscalar-Axialvector
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$C(\delta t)$
0
5
10
15
20
25
$\delta t/a$



naive fit
derivative
$a\,\tilde{f}^{(0)}_{1,8}$
$f_8$
$f_1$
0.2
0.15
0.1
0.05
0
2
3
4
5
$t_{min}/a$

# mixing angles



- normalization: $f_\pi^{(exp)} = \sqrt{2} F_\pi^{(exp)} = 130\,\text{MeV}$
- **U103**: $N_f = 3$, no mixing, $\theta = \theta' = 0$

$$f_8 = f_\pi = 146(2)\,\text{MeV}, \qquad f_1 = 148(11)(2)^{\text{ren}}\,\text{MeV}$$

- **H105**: $N_f = 2 + 1$

$$f_8 = 180(3)\,\text{MeV}, \qquad f_1 = 162(29)(2)^{\text{ren}}\,\text{MeV}$$

$$\theta_8 = -0.54(8) = -31(5)^\circ, \qquad \theta_1 = -0.10(8) = -6(5)^\circ$$

Results preliminary and errors are purely statistical.

# Conclusions and outlook

- Noise reduction techniques allow accessing the $\eta/\eta'$ system, including decay constants.
- Combined fit allows for a direct and controlled way of extracting physical quantities
- However, a understanding of excited states crucial for reliable analysis.
- Going towards finer lattice spacings will alleviate the "window problem"
- CLS ensembles along two different mass trajectories will allow for reliable chiral extrapolation.