Nucleon radii and form factors at $Q^2 = 0$ using momentum derivatives

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Introduction

- Proton radius puzzle: 7σ discrepancy between Muonic hydrogen Lamb shift ($r_E^p = 0.84087(39)$ fm) and atomic hydrogen and scattering experiments using electrons ($r_E^p = 0.8775(51)$ fm)
- Model dependent fit
- $Q^2_{min} < 0.005$ GeV^2 from scattering experiment (e.g. from Mainz, Phys.Rev.Lett. 105 (2010) 242001) whereas on the lattice $Q^2_{min} = 0.05$ GeV^2 for $V = (5.8 \text{ fm})^3$
- Controversy about finding the radius from fitting scattering data.

  "Consistency of electron scattering data with a small proton radius", **Phys.Lett. B737 (2014) 57-59, Phys.Rev. C93 (2016) no.6, 065207, Phys.Rev. C93 (2016) no.5, 055207**

  "Solution of the proton radius puzzle? Low momentum transfer electron scattering data are not enough", **arXiv:1511.00479, Phys.Rev. D92 (2015) no.1, 013013, arXiv:1606.02159**
- This motivates the need for a direct calculation of the radius without fitting to form factors.

Rome method: Momentum derivatives of quark propagator

(Phys. Lett. B 718, 589 (2012) [arXiv:1208.5914]

- On a lattice with finite size and quarks satisfying PBC,

$$C_2(\vec{p},t) = \sum_{\vec{x}} e^{-i\vec{p}\vec{x}}\,\mathrm{Tr}\left[\langle N(\vec{x},t)\bar{N}(0)\rangle\Gamma_{pol}\right] = \sum_{\vec{x}} \mathrm{Tr}\Big[\epsilon^{abc}\epsilon^{def} f_{\alpha\gamma\delta\epsilon} f_{\beta\zeta\eta\theta}$$

$$\Big\langle \underbrace{e^{-i\vec{p}\vec{x}} G^{cd}_{\epsilon\zeta}(x,0)}_{G^{cd}_{\epsilon\zeta}(x,0;\vec{p})} \left(G^{af}_{\gamma\theta}(x,0)G^{be}_{\delta\eta}(x,0) - G^{ae}_{\gamma\eta}(x,0)G^{bf}_{\delta\theta}(x,0)\right)\Big\rangle (\Gamma_{pol})_{\alpha\beta}\Big]$$

- $G(x,0;\vec{p})$ can be obtained by solving the lattice Dirac equation with link variables rescaled by a phase factor,

$$U_k(x) \to e^{ip_k} U_k(x), \qquad \sum_y D(x,y;\vec{p})G(y,z;\vec{p}) = \delta_{x,z}$$

This is equivalent to a lattice propagator associated with quark field satisfying twisted boundary conditions.

- Allow $\vec{p}$ to be continuous and expand the Dirac operator,

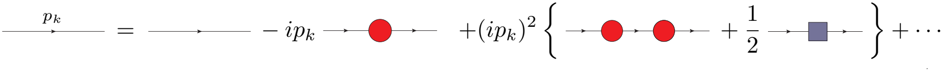
$$D[U,\vec{p}] = D[U] + p_k \frac{\partial D}{\partial p_k}\Big|_{\vec{p}=0} + \frac{p_k^2}{2}\frac{\partial^2 D}{\partial p_k^2}\Big|_{\vec{p}=0} + \dots$$

- Using $DG = 1$ and for clover-improved Wilson action, we find

$$\frac{\partial}{\partial p_k} G(x,y;\vec{p})\Big|_{\vec{p}=0} = -i\sum_z G(x,z)\Gamma_V^k G(z,y)$$

$$\frac{\partial^2}{\partial p_k^2} G(x,y;\vec{p})\Big|_{\vec{p}=0} = -2\sum_{z,z'} G(x,z)\Gamma_V^k G(z,z')\Gamma_V^k G(z',y) - \sum_z G(x,z)\Gamma_T^k G(z,y)$$

$$\Gamma_{V/T}^k G(z,y) \equiv U_k^\dagger(x-\hat{k})\frac{1+\gamma^k}{2}G(z-\hat{k},y) \mp U_k(z)\frac{1-\gamma^k}{2}G(z+\hat{k},y)$$

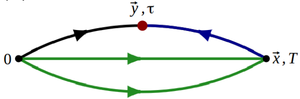


- Extend this to the case of smeared propagators

Momentum derivatives of three-point function

- Three-point function:

$$C_3^\Gamma(\vec{p},\vec{p}',T,\tau) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}'(\vec{x}-\vec{y})} e^{-i\vec{p}\vec{y}}\, \mathrm{Tr}\left[\left\langle N(\vec{x},T) O_\Gamma(\vec{y},\tau) \bar{N}(0) \right\rangle \Gamma_{pol}\right]$$



- Lattice 2016: Momentum derivatives with respect to $\vec{p}$ ($\vec{p}' = 0$)

$$C_3^\Gamma(\vec{p},T,\tau) \sim \sum_{\vec{y}} \mathrm{Tr}\left[\langle G_S(y) \Gamma G(y,0;\vec{p})\rangle\right]$$

$G_S(y)$: The sequential back-ward propagator (independent of $\vec{p}$).

- No additional sequential propagators are needed.
- This approach yields noisy second derivatives which motivates the following alternative implementation.

Momentum derivatives of three-point function

- New method: Momentum derivatives with respect to $\vec{p}$ and $\vec{p}'$

$$C_3^\Gamma(\vec{p}',\vec{p},T,\tau) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}'(\vec{x}-\vec{y})} e^{-i\vec{p}\vec{y}} \left\langle \mathrm{Tr}[S_{\Gamma_{pol}}(0;x)G(x,y)\Gamma G(y,0)] \right\rangle$$
$$= \sum_{\vec{x},\vec{y}} \left\langle \mathrm{Tr}\left[\left(\gamma_5 G(y,x;\vec{p}')\gamma_5 S^\dagger_{\Gamma_{pol}}(0;x) \right)^\dagger \Gamma G(y,0;\vec{p}) \right] \right\rangle$$

$S_{\Gamma_{pol}}(0;x)$: sequential source

- We need the following mixed derivatives

$$\frac{\partial}{\partial p'^j}\frac{\partial}{\partial p^k} C_3^\Gamma((\vec{p}',\vec{p},T,\tau)\Big|_{\vec{p}=\vec{p}'=0} =$$
$$\sum_{\vec{x},\vec{y}} \left\langle \mathrm{Tr}\left[\left(\gamma_5 \frac{\partial}{\partial p'^j} G(y,x;\vec{p}')\Big|_{\vec{p}'=0} \gamma_5 S^\dagger_{q,\Gamma_{pol}}(0;x) \right)^\dagger \Gamma \frac{\partial}{\partial p^k} G(y,0;\vec{p})\Big|_{\vec{p}=0} \right] \right\rangle$$

- Additional sequential propagators are needed for every T.

Ground-state contributions :

$$C_2(\vec{p},t) = e^{-Et}\,\mathrm{Tr}\left[\,\langle\Omega|N|p\rangle\,\langle p|\bar{N}|\Omega\rangle\Gamma_{pol}\right]$$

$$C_3^{\mu}(\vec{p}',\vec{p},T,\tau) = e^{-E'(T-\tau)}\,e^{-E(\vec{p})\tau}\;\mathrm{Tr}\left[\langle\Omega|N|p'\rangle\,\langle p'|\bar{q}\gamma^{\mu}q|p\rangle\,\langle p|\bar{N}|\Omega\rangle\Gamma_{pol}\right]$$

$\langle p'|V_q^{\mu}|p\rangle = \bar{u}(p')\,\underbrace{\mathscr{F}(\Gamma,\vec{p}',\vec{p})}_{F_1^q\,\gamma^{\mu} + F_2^q\,\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}}\,u(p)$, and $\langle\Omega|N|p\rangle = Z(\vec{p})[\Gamma u(p)]$

$$R^{\mu}(\vec{p}',\vec{p}) = \frac{C_3^{\mu}(\vec{p}',\vec{p},T,\tau)}{\sqrt{C_2(\vec{p}',T)\,C_2(\vec{p},T)}}\sqrt{\frac{C_2(\vec{p},T-\tau)\,C_2(\vec{p}',\tau)}{C_2(\vec{p}',T-\tau)\,C_2(\vec{p},\tau)}}$$

$$= \frac{\mathrm{Tr}\left[(m+\not{p}')\mathscr{F}(\gamma^{\mu},\vec{p}',\vec{p})(m+\not{p})\Gamma_{pol}\right]}{16\sqrt{EE'(E+m)(E'+m)}}$$

The derivative method (1)

Lattice 2016: compute first- and second-order momentum derivatives with respect to the initial state

- Compute $\frac{\partial R^\mu}{\partial p_i}\Big|_{\vec{p}=0}$ and $\frac{\partial^2 R^\mu}{\partial {p_i}^2}\Big|_{\vec{p}=0}$ for $\mu = 0, 1, 2$ vector components in x, y and z directions
- For $\Gamma_{\text{pol}} = (1 + \gamma^3\gamma_5)\frac{1+\gamma^0}{2}$:

$$\kappa = -2\, m\ \text{Im}(R^2)' - R^0$$

$$r_1^2 = \frac{12\, m\, \text{Im}(R^2)' + 3\, R^0 - 12\, m^2\, (R^0)''}{4\, m^2\, R^0}$$

Where $r_1^2 = \frac{-6}{F_1}\ \frac{dF_1}{dQ^2}\Big|_{Q^2=0}$

Average over equivalent vector components and directions:

$$(R^2)' = \frac{1}{2}(\partial_1 R^2\ - \partial_2 R^1), \quad (R^0)'' = \frac{1}{3}(\partial_1^2 R^0\ + \partial_2^2 R^0\ + \partial_3^2 R^0\ )$$

The derivative method (2)

- New method: simultaneously compute first-order momentum derivatives with respect to both initial and final state
- Compute:

$$\frac{\frac{\partial}{\partial p'^j}\frac{\partial}{\partial p^j} C_3^0(\vec{p}', \vec{p}, T, \tau)|_{\vec{p}=\vec{p}'=0}}{C_2(\vec{0}, \tau)} = \frac{1}{4m^2}\left[F_1 + 2F_2\right] + \frac{1}{3} F_1 [r_1]^2$$

Removal of excited states

- Ratio-plateau method: compute ratio

$$R(T,\tau) = c_{00} + c_{10}e^{-\Delta E_{10}(\vec{p})\tau} + c_{01}e^{-\Delta E_{10}(\vec{p}')(T-\tau)}$$

  where c_{00} is the desired ground-state matrix element. Then average a fixed number of points around $\tau = T/2$.

  - Lattice 2016: $R,\ \frac{\partial}{\partial p^i}R\Big|_{\vec{p}=0} \sim e^{-\Delta E_{10}\ T/2},\ \frac{\partial^2}{\partial p^{i\,2}}R\Big|_{\vec{p}=0} \sim T\ e^{-\Delta E_{10}T/2}$
  - New method: $\frac{\partial}{\partial p'^j}\frac{\partial}{\partial p^k}R\Big|_{\vec{p}=0} \sim e^{-\Delta E_{10}\ T/2}$

- Summation method: compute sums

$$S(T) = \sum_{\tau} R(T,\tau) = b + c_{00}T + dTe^{-\Delta ET} + \ldots$$

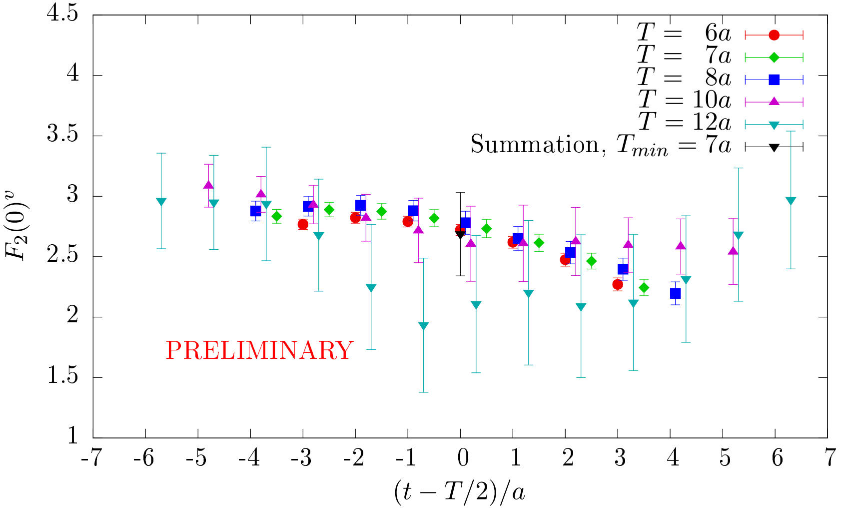
  then find their slope, which gives c_{00}.

  - Lattice 2016: $S,\ \frac{\partial}{\partial p^i}S\Big|_{\vec{p}=0} \sim T\ e^{-\Delta E_{10}T},\ \frac{\partial^2}{\partial p^{i\,2}}S\Big|_{\vec{p}=0} \sim T^2\ e^{-\Delta E_{10}T}$
  - New method: $\frac{\partial}{\partial p'^j}\frac{\partial}{\partial p^k}S\Big|_{\vec{p}=\vec{p}'=0} \sim T\ e^{-\Delta E_{10}T}$

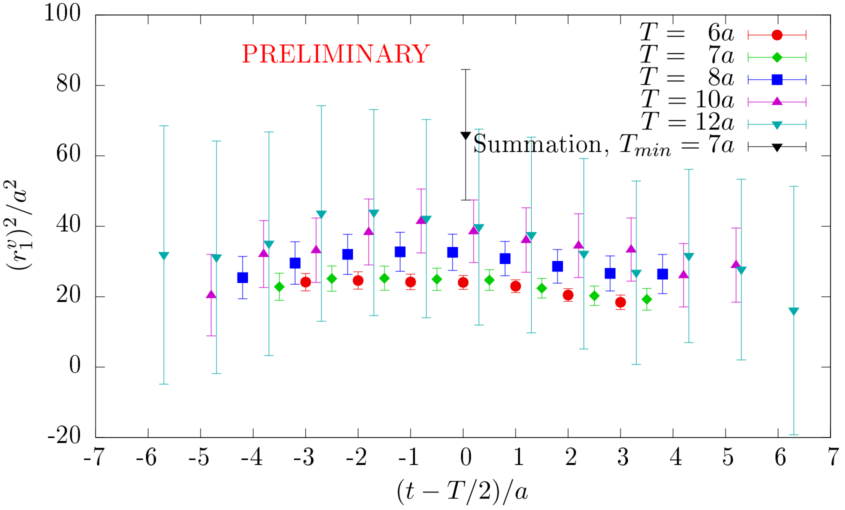
Ensemble

- BMW $N_f = 2 + 1$ 2HEX-clover, $L_s = L_t = 48$.
- $\beta = 3.31$, $a^{-1} = 1.697$ GeV, $a = 0.116$ fm .
- Physical pion mass, $m_\pi L = 4$.
- Five source-sink separations $T/a \in \{6, 7, 8, 10, 12\}$, which corresponds to T between 0.7 and 1.4 fm.
- 212 gauge configurations analysed
- AMA, using 96 sources per configuration with approximate propagators and one source for the bias correction.

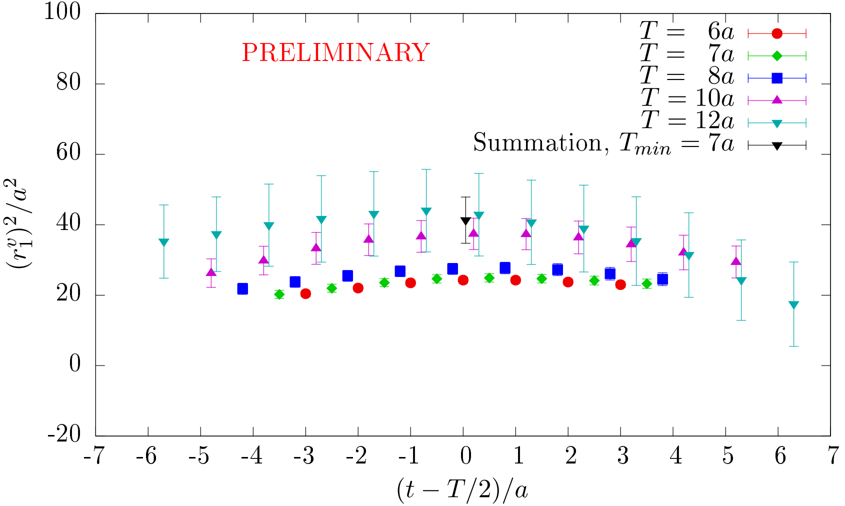
Isovector anomalous magnetic moment

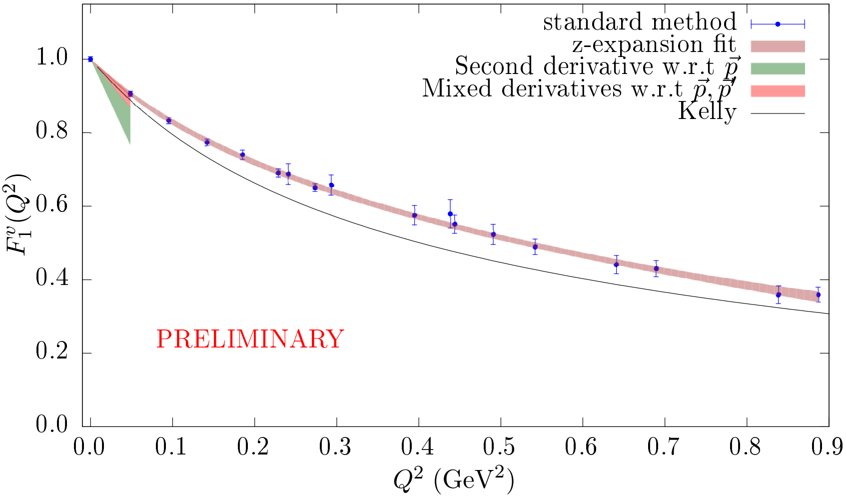


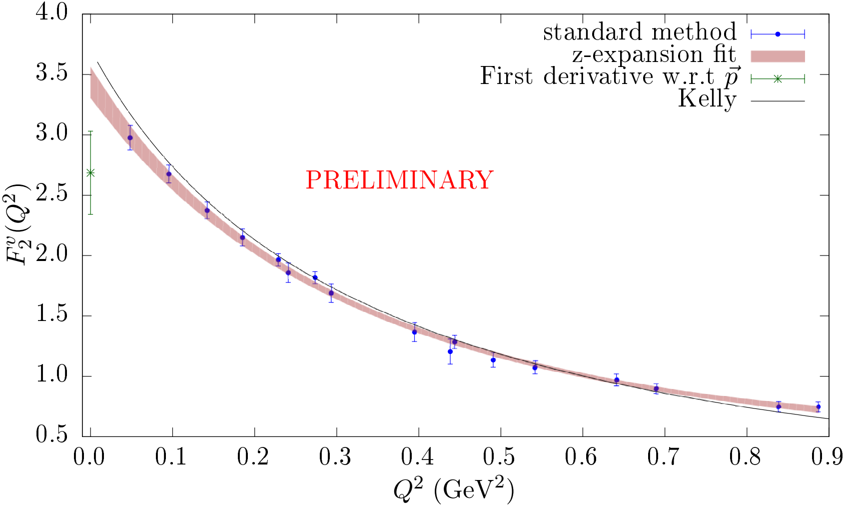
The Dirac radius (second momentum derivative w.r.t $\vec{p}$)



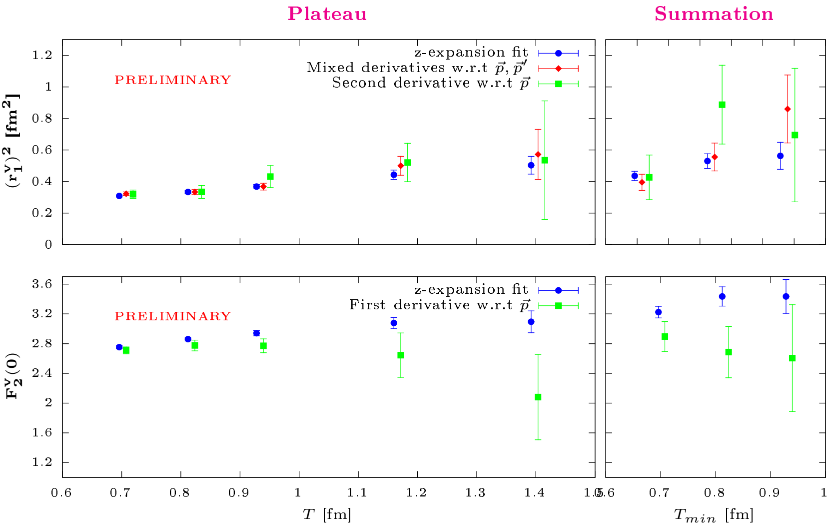
The Dirac radius (mixed momentum derivatives w.r.t $\vec{p}, \vec{p}'$)




standard method
z-expansion fit
Second derivative w.r.t $\vec{p}$
Mixed derivatives w.r.t $\vec{p}, \vec{p}'$
Kelly
PRELIMINARY
$F_1^v(Q^2)$
Q^2 (GeV^2)
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.3
0.5
0.7
0.9


standard method
z-expansion fit
First derivative w.r.t $\vec{p}$
Kelly
PRELIMINARY
$F_2^v(Q^2)$
Q^2 (GeV^2)
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Summary plot



Summary

- Our approach is based on the Rome method and is model independent.
- Using the mixed derivative approach, we were able to reduce the statistical uncertainity on $[r_1^v]^2$ and obtain results consistent with fitting to form factors.
- We also extended this method to the axial radius and form factor at zero Q^2.
- This method may help lead to reliable lattice calculations of the proton radius and to better understanding of the radius puzzle.