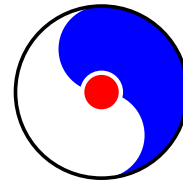


Finite volume study for muon $g-2$ light-by-light contribution

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RIKEN BNL
Research Center

- Motivation
- Finite Volume Effect in QCD+QED
- QCD box in QED box
- Infinite Volume Photon and Lepton
- Current conservation and subtraction
- Results



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Introduction

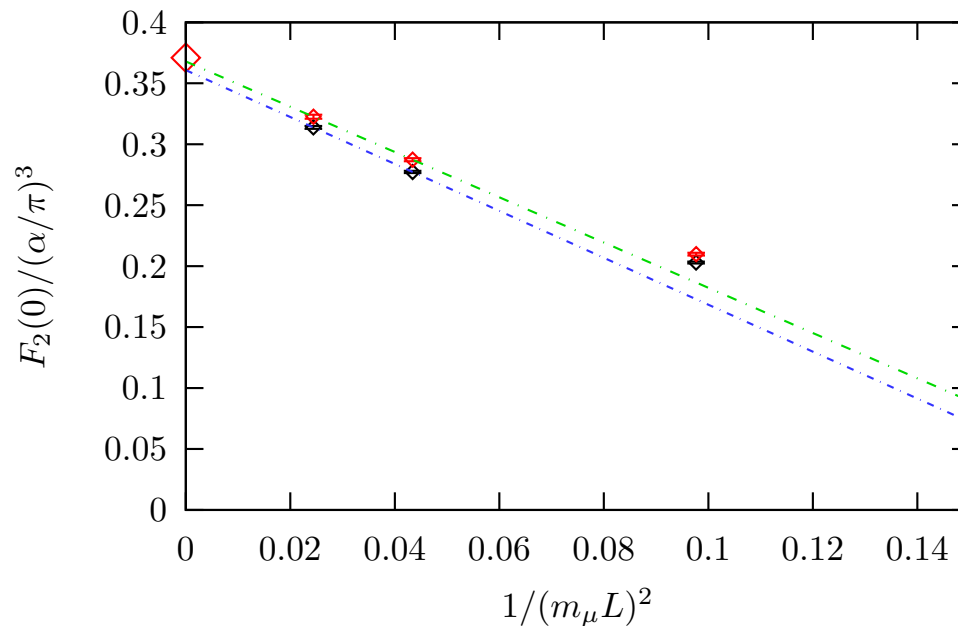
- Systematic errors of Hadronic contribution for $(g - 2)_\mu$
 - **Finite Volume** : $1/L^2 \rightarrow \exp(m_\pi L)$
 - **Discretization error** : a^2 scaling from $a^{-1}=1.73$ GeV, 48^3 and $a^{-1}=2.38$ GeV 64^3
 - Disconnected diagram : $SU(3)_F$ leading diagram. We need to check size of other disconnected diagrams
 - Isospin breaking ?
 - ...
- c.f. In HVP case : Using Time Moment Representation (TMR), Finite Volume error : 2-7% at $m_\pi L \sim 4$ [E. Shintani, Thur. 17:10]

Finite Volume Error in QCD+QED

- Photon is massless. Muon, $m_\mu=106 \text{ MeV} < m_\pi$.
- We have used QED_L [2008, Hayakawa Uno] for previous LbL calculation : $\mathcal{O}(1/L^2)$ error
- Lepton LbL (pure QED, loop mass is also m_μ) [L. Jin et al. 1510.07100] :

$$[F_2(0)]_{\text{quad}} / (\alpha/\pi)^3 = 0.3679(42) - 1.86(11)/(m_\mu L)^2,$$

$$[F_2(0)]_{\text{PT}} / (\alpha/\pi)^3 = 0.3710052921,$$



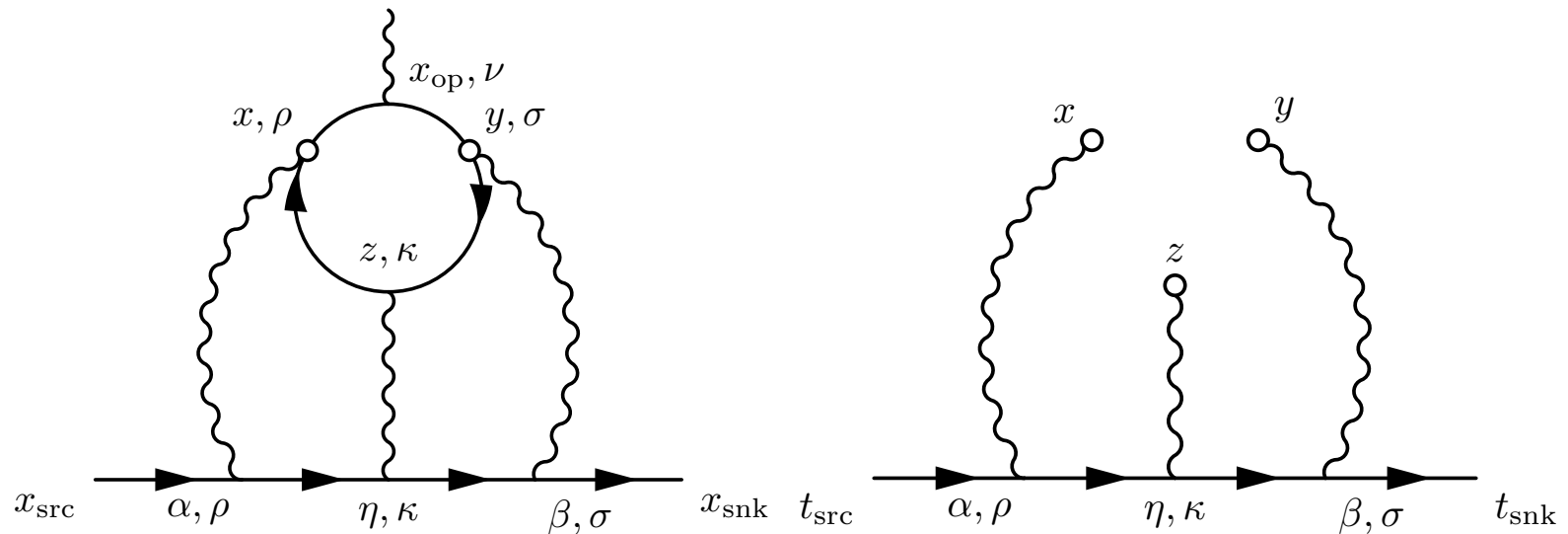
HLbL point source method [L. Jin et al. 1510.07100]

- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \rightarrow 0$ limit

$$\frac{F_2^{\text{cHLbL}}(q^2 = 0) (\sigma_{s',s})_i}{m} \frac{1}{2} = \frac{\sum_{x,y,z,x_{\text{op}}} \epsilon_{i,j,k} (x_{\text{op}} - x_{\text{ref}})_j \cdot i \bar{u}_{s'}(\vec{0}) \mathcal{F}_k^C(x, y, z, x_{\text{op}}) u_s(\vec{0})}{2VT}$$

- Stochastic sampling of x and y point pairs. Sum over x and z .

$$\mathcal{F}_\nu^C(x, y, z, x_{\text{op}}) = (-ie)^6 \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}}),$$



Finite Volume error of HLbL in QED_L [L. Jin 1509.08372]

- Massless photon propagator $\frac{1}{|x-y|^2}$
- One of three internal photons of $\mathcal{F}_\nu^C(x, y, z, x_{\text{op}})$:

$$\begin{aligned} G(\mathbf{k}, t_2, t_1) &= \int \frac{dp_0}{2\pi} e^{ip_0(t_2-t_1)} \frac{1}{p_0^2 + \mathbf{k}^2} \\ &= \frac{1}{2|\mathbf{k}|} \exp(-|\mathbf{k}||t_2 - t_1|). \end{aligned}$$

- For a single internal photon, the behavior of the integrand in small \mathbf{k} region is roughly

$$\int_{-\infty}^{\infty} dt_{\text{line}} \frac{1}{|\mathbf{k}|} \exp(-|\mathbf{k}||t_{\text{loop}} - t_{\text{line}}|) |\mathbf{k}| \sim \mathcal{O}\left(\frac{1}{|\mathbf{k}|}\right),$$

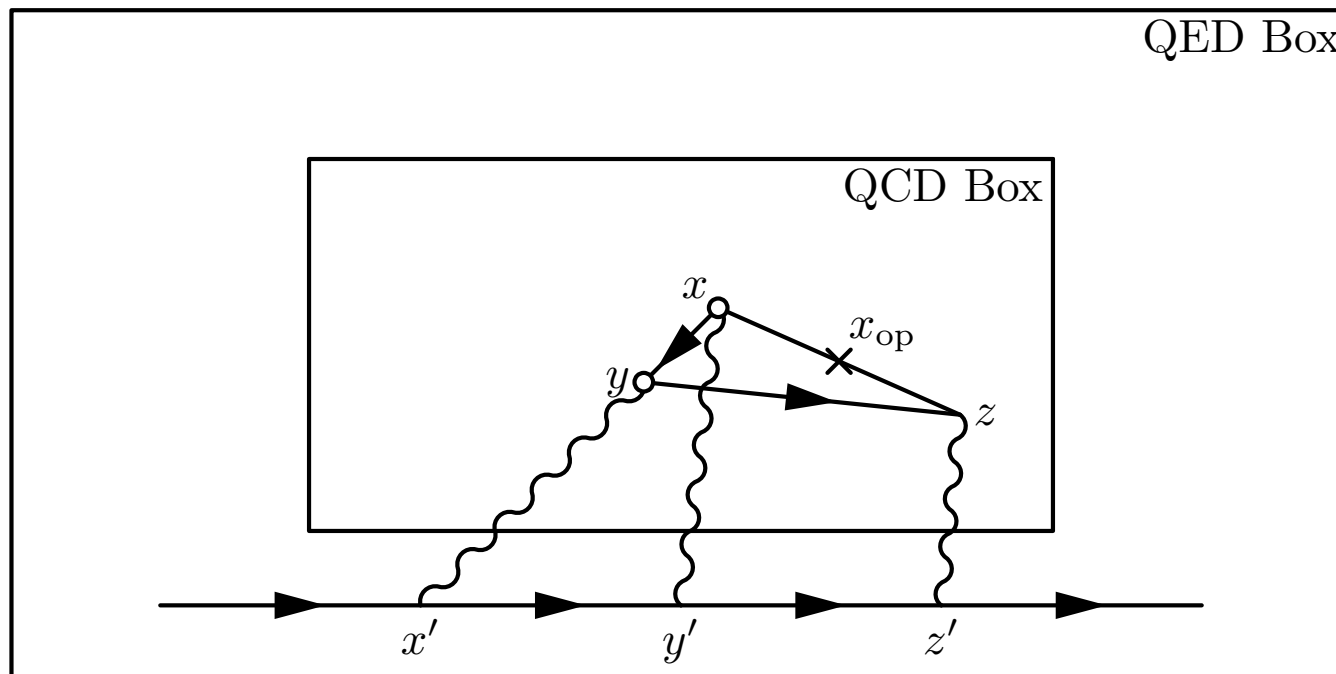
(t_{line} : time of EM vertex on external line, t_{loop} the location in time of the internal muon loop) The last factor of $|\mathbf{k}|$ comes from the fact that the photon has to couple to a neutral loop and the coupling at such a small momentum photon is suppressed by a factor of $|\mathbf{k}|$.

- Finite Volume correction should be proportion to

$$\int_0^{1/L} \mathcal{O}\left(\frac{1}{|\mathbf{k}|}\right) d^3k \sim \mathcal{O}\left(\frac{1}{L^2}\right).$$

QCD box inside a large QED box [L. Jin et al. 1511.05198]

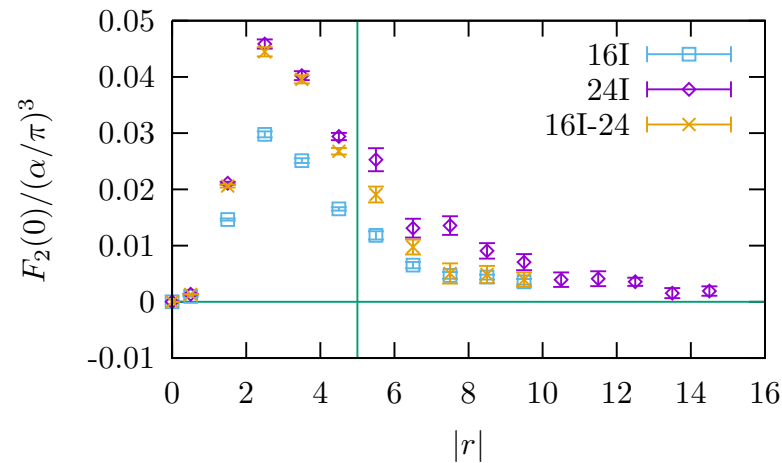
- One could use a larger 4D QED box to compute photon+lepton part of the diagram $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$, and compute the hadron part $\mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}})$ in a smaller QCD box.
- Hadron part could be recycled for different size of QED box, which introduces a beneficial correlation in taking $V \rightarrow \infty$ limit.
- As far as the integral outside of QCD box is small (from $\exp(-m_\pi r)$ suppression of hadron 4pt of size r), $\mathcal{O}(1/L^2)$ error would be largely suppressed.



QCD box inside a large QED box [L. Jin et al. 1511.05198]

- Result using $a^{-1} = 1.747\text{GeV}$, $m_\pi = 423\text{MeV}$, $m_\mu = 332\text{MeV}$.

Ensemble	QCD Size	QED Size	$t_{\text{snk}} - t_{\text{src}}$	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	$16^3 \times 32$	$16^3 \times 32$	16	0.1158(8)
24I	$24^3 \times 64$	$24^3 \times 64$	32	0.2144(27)
16I-24	$16^3 \times 32$	$24^3 \times 64$	32	0.1674(22)



- By using larger QED box, the major part of the finite-volume effects have been removed. But visible disagreement from QCD FV error on $(1.8\text{ fm})^3$ volume.
- Integrand plot ($|r| = |x - y|$). For small r , 16I and 16I-24 is very close, but visible deviation for $r > 5$, close to $L/2=8$.

Infinite Volume Photon and Lepton QED_∞

[Feynman, Schwinger, Tomonaga]

- Instead of, or, in addition to, larger QED box, one could use infinite volume QED to compute $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$.
- Hadron part $\mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{op})$ has following features due to the mass gap :
 - ▷ For large distance separation, the 4pt Green function is exponentially suppressed: $\mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{op}) \sim \exp[-m_\pi \times \text{dist}(x, y, z, x_{op})]$
 - ▷ For fixed (x, y, z, x_{op}) , FV error (wraparound effect etc.) is exponentially suppressed: $\mathcal{H}_{\rho,\sigma,\kappa,\nu}^C|_V - \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C|_\infty \sim \exp[-m_\pi \times L]$
- By using QED_∞ weight function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$, which is not exponentially growing, asymptotic FV correction is exponentially suppressed

$$\Delta_V \left[\sum_{x,y,z,x_{op}} \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{op}) \right] \sim \exp[-m_\pi L]$$

($x_{\text{ref}} = (x + y)/2$ is at middle of QCD box using translational invariance)

- The idea of $(\text{QED})_\infty$ for suppressing FV error was discussed several times e.g. [C. Lehner, TI Lattice 2014] [L. Jin et al CIPAN15, 1509.08372] . More concrete proposals with numerical demonstration for π^0 exchange model by Mainz group [PRL 115(22):222003, 2015, LATTICE2016:164, 2016] .
- For g-2 HVP, infinite QED was used from the beginning [T. Blum, PRL91, 052001, 2003]
- Exponentially suppressed FV error using QED_∞ would be **generic for Euclidean Green's function** (e.g. QED correction to HVP)
- But not necessarily always the case for other quantities (e.g. QED mass shift is being checked [M. Hayakawa]).

$$\frac{\Delta_V M[\text{QED}_L]}{M} \sim -Q^2 \alpha \frac{\kappa}{2ML} \left(1 + \frac{2}{ML} \right)$$

- In addition to the point-source sampling method, QED_∞ may be implemented in the non-perturbative QCD+QED method, or RM123 method.

QED_∞ weight function

$$i^3 \mathcal{G}_{\rho, \sigma, \kappa}(x, y, z) = \mathfrak{G}_{\rho, \sigma, \kappa}(x, y, z) + \mathfrak{G}_{\sigma, \kappa, \rho}(y, z, x) + \text{other 4 permutations.}$$

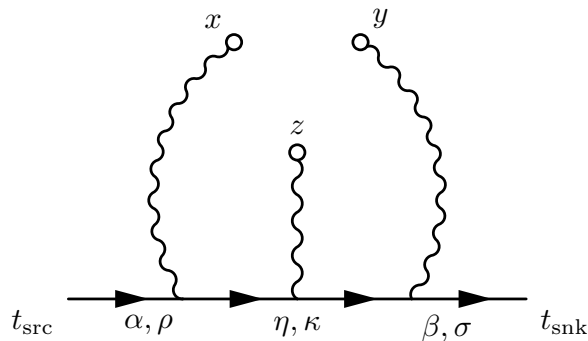
- Take hermitian part and using permutation,

$$\mathfrak{G}_{\rho, \sigma, \kappa}^{(1)}(x, y, z) = \frac{1}{2} \mathfrak{G}_{\rho, \sigma, \kappa}(x, y, z) + \frac{1}{2} [\mathfrak{G}_{\kappa, \sigma, \rho}(z, y, x)]^\dagger$$

gives same F_2 but **infrared finite**.

- In $m_\mu = 1$ unit,

$$\begin{aligned} \mathfrak{G}_{\sigma, \kappa, \rho}^{(1)}(y, z, x) &= \frac{\gamma_0 + 1}{2} i\gamma_\sigma (-\not{\partial}_y + \gamma_0 + 1) i\gamma_\kappa (\not{\partial}_x + \gamma_0 + 1) i\gamma_\rho \frac{\gamma_0 + 1}{2} \\ &\times \frac{1}{4\pi^2} \int d^4\eta \frac{1}{(\eta - z)^2} f(\eta - y) f(x - \eta). \end{aligned}$$



Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0) \rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\text{op}}) \rangle = 0$$

$$\sum_z \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}}) = 0$$

at $V \rightarrow \infty$ and $a \rightarrow 0$ limit (we use local currents).

- We could further change QED weight

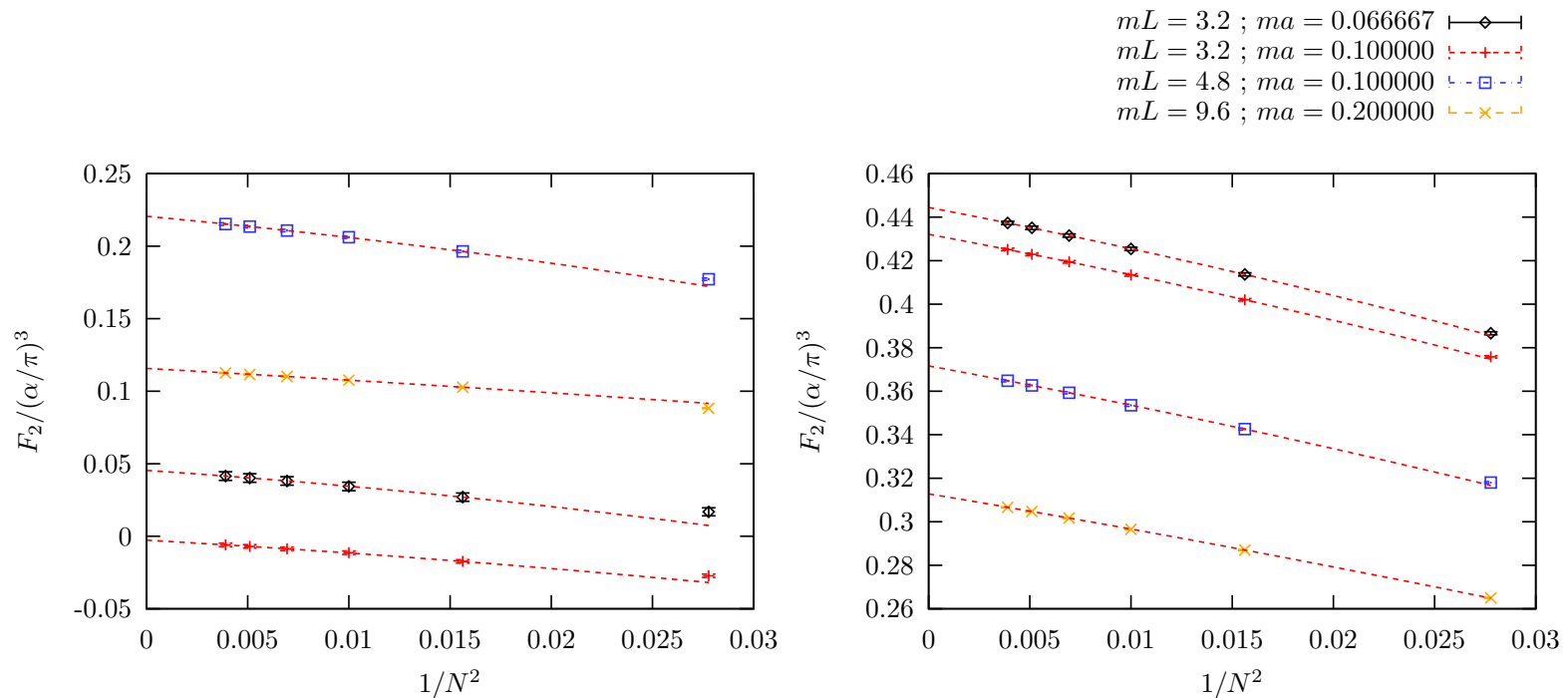
$$\mathfrak{G}_{\rho,\sigma,\kappa}^{(2)}(x, y, z) = \mathfrak{G}_{\rho,\sigma,\kappa}^{(1)}(x, y, z) - \mathfrak{G}_{\rho,\sigma,\kappa}^{(1)}(y, y, z) - \mathfrak{G}_{\rho,\sigma,\kappa}^{(1)}(x, y, y) + \mathfrak{G}_{\rho,\sigma,\kappa}^{(1)}(y, y, y)$$

without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}})$.

- Subtraction changes **discretization error** and **finite volume error**.
- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.
- Also now $\mathfrak{G}_{\sigma,\kappa,\rho}^{(2)}(z, z, x) = \mathfrak{G}_{\sigma,\kappa,\rho}^{(2)}(y, z, z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x, y, z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x - y| < 11$ fm).

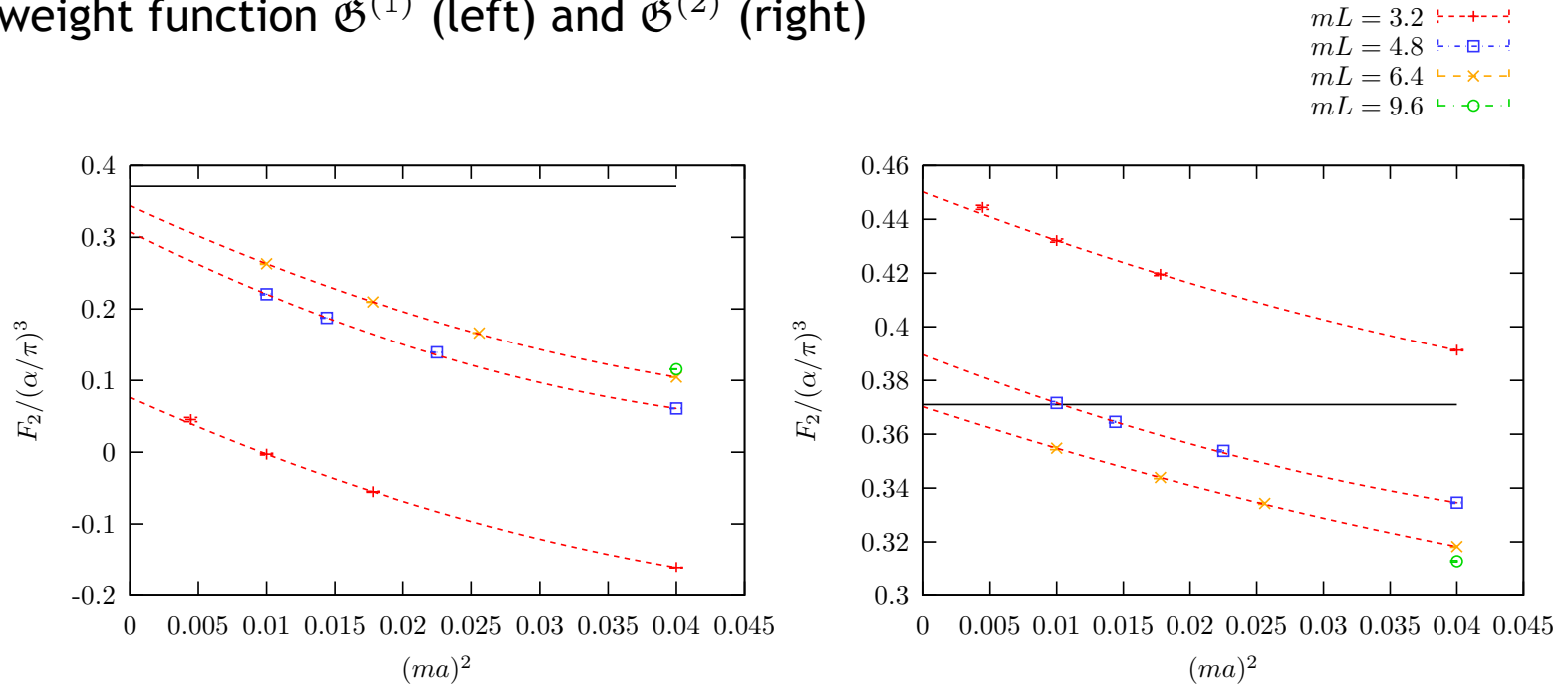
Results, QED case, Grid interpolation

- $m_{\text{loop}} = m_{\mu}$
- QED weight function without and with current-conservation subtraction $\mathfrak{G}^{(1)}$ (left) and $\mathfrak{G}^{(2)}$ (right)
- Grid points for (x, y, z) interpolation : $N = 6, 8, 10, 12, 14, 16$, second-order fits to $1/N^2$



Results, QED case, Discretization error

- QED weight function $\mathcal{G}^{(1)}$ (left) and $\mathcal{G}^{(2)}$ (right)

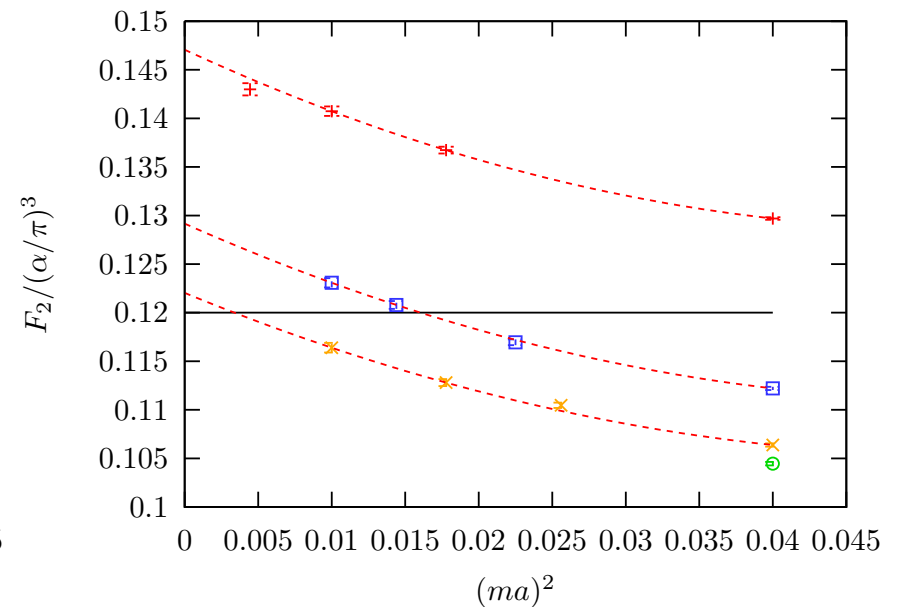
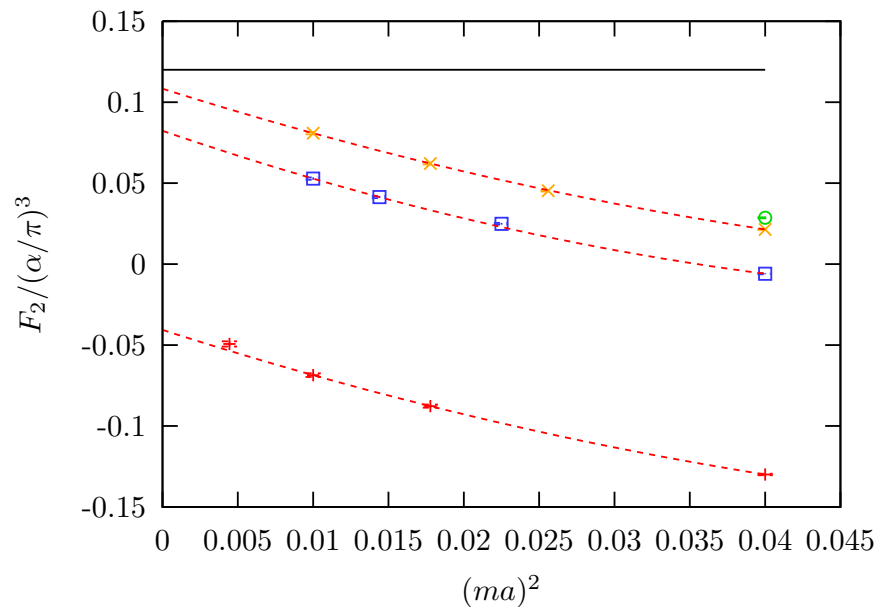


- Discretization error becomes almost independent of volume when $mL \geq 4.8$;
- The finite volume effect becomes negligible for $mL = 9.6$.
- Fit data with $mL \geq 4.8$, (and 3rd order also for systematic error)

$$F_2(L, a) = F_2(L) + k_1 a^2 + k_2 a^4.$$

Results, QED case, Discretization error, $m_{\text{loop}} = 2m_\mu$

- QED weight function without and with current-conservation subtraction $\mathcal{G}^{(1)}$ (left) and $\mathcal{G}^{(2)}$ (right)

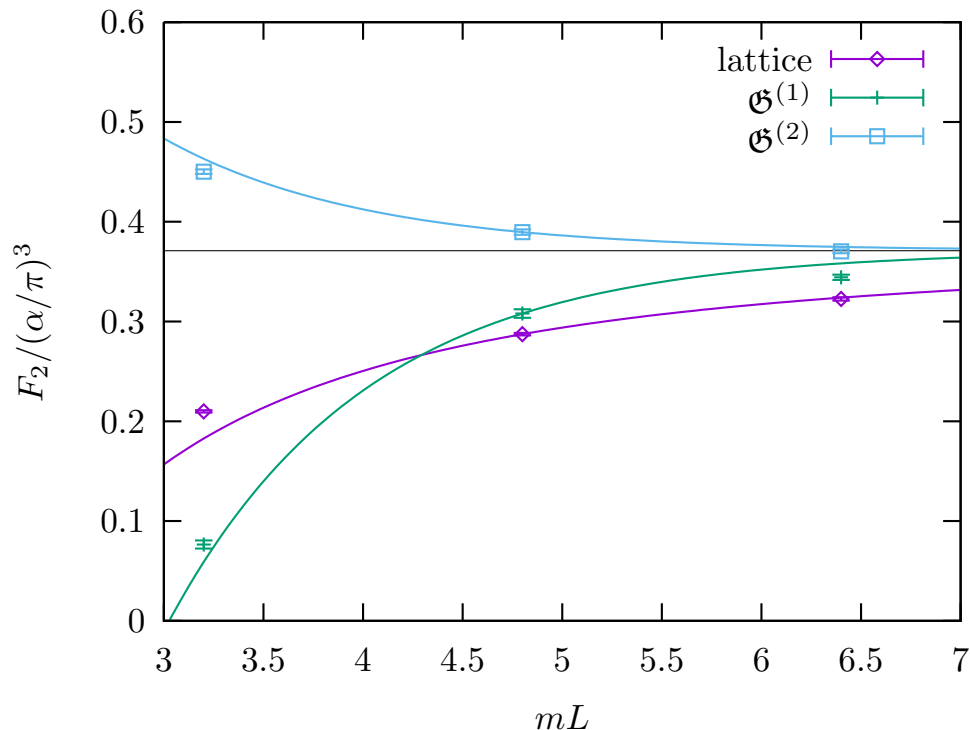


- $V \rightarrow \infty$ values are taken from $mL = 9.6$ case : $\lim_{L \rightarrow \infty} F_2(L) \approx F_2(9.6/m)$.
- For $m_{\text{loop}}/m_\mu = 1$ and 2,

$$F_2/(\alpha/\pi)^3 = 0.3686(37)(35), 0.1232(30)(28),$$

QED perturbation results : 0.371, 0.120

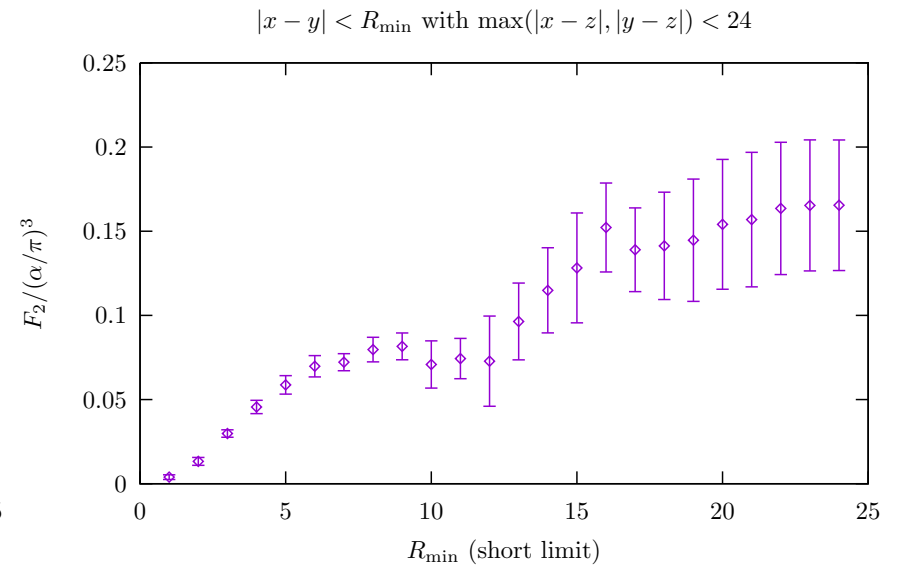
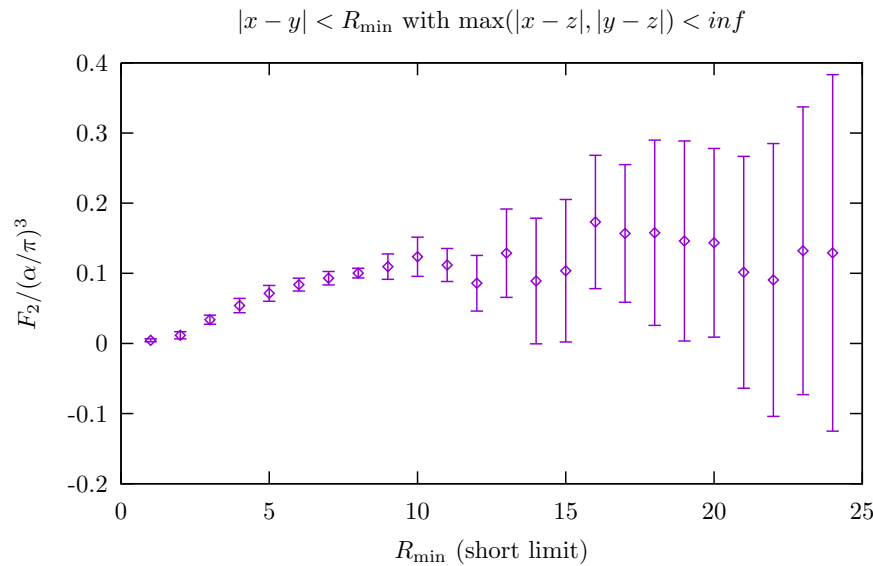
Results, QED case, Finite Volume Error



- QED weight : QED_L (purple diamond), QED_∞ without subtraction (green plus), with subtraction (blue square)
- Curves correspond to expected finite volume scaling ($0.371 + k/L^2$) and infinite volume scaling ($0.371 + ke^{-mL}$), where the coefficient k is chosen to match the data at $mL = 4.8$.
- The right most point for the finite volume weighting function lies a bit off its scaling curve because the discretization error has not been completely removed, and the coefficient k does not contain any possible volume dependence.

Preliminary results, QCD case

- QCD case with physical point quark mass,
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73$ GeV, $m_\pi = 139$ MeV, $m_\mu = 106$ MeV.



- c.f. QED_L case, $\left. \frac{g_{\mu-2}}{2} \right|_{\text{cHLbL}} = (0.0926 \pm 0.0077) \left(\frac{\alpha}{\pi}\right)^3$

Conclusion and Discussion

- Finite Volume Error in HLbL calculation is investigated.
- QED_∞ weight function, infinite volume/continuum limit diagram, is shown to be advantageous in suppressing finite volume error (improved from $\mathcal{O}(1/L^2)$).

$$F_2(L, a) = F_2 + \mathcal{O}(e^{-mL}) + \mathcal{O}((ma)^2).$$

- Exponential suppression of FV would be a generic feature of Euclidean Green's function using QED_∞ .
- QED weight function with subtraction, $\mathfrak{G}^{(2)}$, using EM current conservation, is useful for (similar method already used in HVP) leads smaller finite volume and discretization error error. (similar observation for FV in HVP case).
- Very preliminary QCD results with $\mathfrak{G}^{(2)}$ is shown.
- Further study for $\mathfrak{G}^{(2)}$ effect in π^0 exchange model would be interesting to forecast QCD's finite size effect (on going)
- $1/a = 1$ GeV coarse DWF ensemble at physical point with 3 volumes, $(4.8 \text{ fm})^3$, $(6.4 \text{ fm})^3$, and $(9.6 \text{ fm})^3$ will be very useful. [R. Mawhinney's talk Tuesday]