A new method for the beta function in the chiral symmetry broken phase

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lattice simulations dedicated to beta function measurements are feasible but expensive — often work directly at massless fermion limit, with some choice of boundary conditions

complementary work to studies of the p- and/or epsilon-regimes, with their own independent and costly simulations — but does the renormalized coupling reach the relevant range?

could the beta function be measured directly via p-regime simulations?
— “cheap” reusing of the same ensembles
— determination at the relevant renormalized coupling
— consistency with other beta function methods

Yes: will show results for the sextet SU(3) model at $g^2 = 6.7$
gradient flow of gauge fields

\[ \frac{dA_\mu}{dt} = D_\nu F_{\nu\mu} \]

flow “time” \( t \)

\[ A_\mu(t), \quad E = \frac{1}{4}(F_{\mu\nu}^a)^2 \]

perturbation theory

\[ \overline{\text{MS}}: \langle E \rangle = \frac{3(N^2 - 1)g^2}{128\pi^2t^2} \left\{ 1 + c_1g^2 + \mathcal{O}(g^4) \right\} \quad \text{SU}(N) \]

RG scale \( \mu = 1/\sqrt{8t} \)

non-perturbative definition

\[ g^2(t) \equiv \frac{1}{N} \left( \frac{128\pi^2}{3(N^2 - 1)} \right) t^2 \langle E \rangle_{\text{latt}} \]

normalization factor \( \mathcal{N} \)

e.g. boundary conditions

goal: determine \( \mu^2 \frac{dg^2}{d\mu^2} \) at targeted values of renormalized coupling \( g^2 \)

finite or infinite volume? infinitesimal or discrete step? \( g^2(t') - g^2(t) \)

one approach: **step scaling** use finite volume to your advantage

physical volume \( L \) lattice volume \( L/a \)

adjust flow time to lattice volume such that \( c = \sqrt{8t}/L \) held fixed

RG scale is now \( \mu = 1/(cL) \)
in practice e.g. target \( g_c^2(L/a) = 6, \ c = 0.25 \) directly at zero fermion mass

choose lattice volume e.g. \( L/a = 16 \) adjust bare coupling (lattice spacing) until target reached - tuning

next: keep bare coupling (lattice spacing) fixed, change lattice volume, \( c \) held fixed i.e. change RG scale

\[
\beta(g_c^2) = \frac{g_c^2(sL/a) - g_c^2(L/a)}{\log(s^2)}
\]

e.g. \( s = 2 \)

repeat for sequence of lattice volumes e.g.
\( L/a = 16, 18, 20, 24, 28 \rightarrow 2L/a = 32, 36, 40, 48, 56 \)

tune bare coupling such that each satisfies \( g_c^2(L/a) = 6, \ c = 0.25 \)

continuum limit of \( \beta(g_c^2) \) taking \( a/L \rightarrow 0 \)

ESSENTIAL AND EXPENSIVE
alternative: work in infinite volume limit, at non-zero fermion mass

control of flow: can measure directly not only $g^2(t)$ but also $dg^2(t)/dt$

goal: measure $t \cdot dg^2/dt = -\mu^2 \cdot dg^2/d\mu^2$ on p-regime ensembles at targeted values of $g^2$

will require infinite volume extrapolation, followed by chiral extrapolation — have theoretical guidance

continuum limit: lattice scale $t_0/a^2$ set by fixed choice of renormalized coupling e.g.

$$g^2(L/a, t_0/a^2) = 6.7$$

this does not correspond to Lüscher choice $t^2 \langle E \rangle |_{t_0} = 0.3$

once $t_0/a^2$ known, measure $\beta(g^2) = t \cdot (dg^2/dt)|_{t_0}$

infinite volume and chiral limit extrapolations required for both $t_0/a^2$ and $\beta$

continuum limit: $a^2/t_0 \rightarrow 0$

benefits:
— gauge configuration ensembles already available from extensive mass spectrum simulations
— the possible renormalized coupling values are directly related to the correct phase of the theory
— complements step-scaling method of measuring the beta function
will show this for the two flavor two-index symmetric (sextet) SU(3) model

method is general, can apply to other gauge theories

target: \( g^2(L/a, t_0/a^2) = 6.7 \)

must be attainable across all ensembles

bare coupling \( \frac{6}{g_0^2} = 3.20 \) \( L \) coarsest lattice spacing

SSC:
- simulation gauge action: Symanzik
- flow gauge action: Symanzik
- discretization of \( E \) : Clover

staggered fermions, 2 steps of stout improvement

for this particular ensemble: \( 56^3 \times 96, \ ma = 0.001 \)

\[ t_0/a^2 = 5.531 \pm 0.0915 \]

error in scale determination increases along flow, but don’t want flow too short either - cutoff effects
same ensemble: measure derivative at \( t_0/a^2 = 5.531 \pm 0.0915 \)

approximate derivative
\[
\frac{1}{12\epsilon} \left\{ -F(t + 2\epsilon) + 8F(t + \epsilon) - 8F(t - \epsilon) + F(t - 2\epsilon) \right\} = \frac{dF}{dt}\bigg|_t + \mathcal{O}(\epsilon)^4
\]

same ensemble as before:
\[56^3 \times 96, \quad ma = 0.001\]

accurately determined
\[2t \left( \frac{dg^2}{dt} \right) \bigg|_{t_0} = 1.4939 \pm 0.036\]

repeat for a sequence of ensembles at the same lattice spacing, and the same fermion mass, but different lattice volumes

\[
\beta(t) = 2 \cdot t \frac{dg^2(t)}{dt}
\]

\[56^3 \times 96, \quad \beta = 3.20, \quad m = 0.0010\]

\[\beta = 1.4939 \pm 0.036\]

target \( g^2 = 6.7 \)
infinite volume limit for Goldstone boson at this lattice spacing and fermion mass

finite-volume corrections from Goldstone boson wrapping around the finite lattice

functional form: \( g_1(M_\pi \cdot L, \eta) \)

infinite sum of Bessel functions with aspect ratio \( \eta = L_t/L_s \)

result: infinite volume mass \( M_\pi \cdot a = 0.08154 \pm 0.00015 \)

this will be used as input for infinite volume extrapolations of \( t_0/a^2 \) and \( t \cdot (dg^2/dt)|_{t_0} \)

want footprint of flow smaller than Goldstone boson correlation length \( \sqrt{8t_0/a} \ll 1/(M_\pi \cdot a) \)

influences choice of target \( g^2(L/a, t_0/a^2) \)
same functional form \( g_1(M_\pi \cdot L, \eta) \)
gives infinite volume values at one lattice spacing and one fermion mass
next step: repeat at same lattice spacing and several fermion masses
generate a set of \( t_0(M_\pi^2) \) and corresponding \( \beta(M_\pi^2) \)
how to extrapolate to zero Goldstone boson mass?
chiral expansion for quantities related to gradient flow, such as the scale

\[ t_0 = t_{0,\text{ch}} \left( 1 + k_1 \frac{M^2_\pi}{(4\pi f)^2} + k_2 \frac{M^4_\pi}{(4\pi f)^4} \log \left( \frac{M^2_\pi}{\mu^2} \right) + k_3 \frac{M^4_\pi}{(4\pi f)^4} \right) \]

expected next-to-leading order term is linear in \( M^2_\pi \)

look for this behavior in infinite-volume results

we do not attempt to fit chiral logarithms or \( \mathcal{O}(M^4_\pi) \) terms
linear in $M_{\pi}^2$ behavior visible in both $t_0(M_{\pi}^2)$ and $\beta(M_{\pi}^2)$ at the lightest masses

obtain chiral limit values at one lattice spacing — repeat entire procedure at other lattice spacings with same target $g^2(L/a, t_0/a^2) = 6.7$

results above from the coarsest lattice spacing
alternative: chiral extrapolation directly in fermion mass

linear in $M_\pi^2$ corresponds to linear in $m$

above: results at same coarsest lattice spacing

also a good description of the data

rest of talk: will use this fitting method
these results correspond to a bare coupling \( \frac{6}{g_0^2} = 3.25 \) intermediate lattice spacing

lattice volumes: \( 32^3 \times 64, \ 40^3 \times 80, \ 48^3 \times 96, \ 56^3 \times 96, \ 64^3 \times 96 \)

larger value of \( t_0/a^2 \) in chiral limit corresponding to smaller lattice spacing

\(~15\%\) change in chiral limit of \( \beta(m) \) compared to coarser lattice spacing — modest cutoff effect
third set of results corresponding to bare coupling \( \frac{6}{g_0^2} = 3.30 \) finest lattice spacing

lattice volumes: \( 40^3 \times 80, \ 48^3 \times 96, \ 56^3 \times 96, \ 64^3 \times 96 \)

from chiral limit scale \( t_0/a^2 \) lattice spacing changes by factor \( \sim 1.6 \) from coarsest to finest ensembles

at each lattice spacing, repeated the full procedure: infinite volume limit, then chiral extrapolation
last step: continuum extrapolation

expect cutoff effects are $O(a^2)$

linear extrapolation in $a^2/t_0$

cutoff effects apparently mild

errors in both $t_0/a^2$ and $\beta(g^2, a^2/t_0)$

included in continuum extrapolation

result for beta function at renormalized coupling $g^2 = 6.7$

$$t \cdot \frac{dg^2}{dt} = -\mu^2 \cdot \frac{dg^2}{d\mu^2} = 0.485 \pm 0.030$$

with infinite volume, $M_\pi \cdot a \to 0$, $a^2/t_0 \to 0$ limits
comparison with step-scaling determination of the beta function for the 2 flavor sextet SU(3) model

consistency between **infinitesimal** and **finite-step s=3/2** beta functions

**approach to the conformal window**

- fund $N_f = 4$, $c = 3/10$, $s = 3/2$
- fund $N_f = 8$, $c = 3/10$, $s = 3/2$
- sextet $N_f = 2$, $c = 7/20$, $s = 3/2$
- fund $N_f = 12$, $c = 1/5$, $s = 2$

$m_\sigma/F \sim 6$
SU(4) PNGB

$m_\sigma/F \sim 4$
LSD
SU(4)+4 PNGB

$m_\sigma/F \sim 2-3$

SU(4)+8 PNGB

result at $g^2 = 6.7$

beta function small but non-zero

no more “room” for an Infra Red Fixed Point corresponding to a zero of the beta function

— bridged the gap between perturbative and strong coupling

**result: sextet model appears near-conformal, remains interesting as composite Higgs BSM theory**
Thank you for your attention