Constrained Hybrid Monte Carlo on Multiscale Lattices

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• Actions with different lattice spacings give identical low energy physics if they are related by an RG blocking,

\[ e^{-S^b_c[U_c]} \equiv \int [dU_f] e^{-S_f[U_f]} G[U_c, U_f] \]

• Formally once given a fine action \( S_f[U_f] \) and a blocking kernel \( G[U_c, U_f] \), we can produce the blocked coarse action \( S^b_c[U_c] \).
• The issue is this \( S^b_c[U_c] \) has to be local and simple enough such that it can be handled numerically.
• Given an exact form for $S^b_c[U_c]$, observables can be calculated from constrained fine lattice Monte Carlos in an ensemble of background coarse lattice links.

\[
\langle O \rangle = \frac{\int [dU_f] e^{-S_f[U_f]} O[U_f]}{\int [dU_f] e^{-S_f[U_f]}}
\]

\[
= \frac{\int [dU_c] G[U_c, U_f] [dU_f] e^{-S_f[U_f]} O[U_f]}{\int [dU_c] G[U_c, U_f] [dU_f] e^{-S_f[U_f]}}
\]

\[
= \frac{\int [dU_c] e^{-S^b_c[U_c]} \int [dU_f] e^{-S_f[U_f]} G[U_c, U_f] O[U_f]}{\int [dU_f] e^{-S_f[U_f]} G[U_c, U_f]}
\]

\[
= \frac{\int [dU_c] e^{-S^b_c[U_c]}}{\int [dU_c] e^{-S^b_c[U_c]}}
\]

• This is ambitious. We will only focus on the question of how to obtain $S^b_c[U_c]$, for now.
• Recent $2 + 1$ flavor ensembles generated with ID and MDWF action by RBC/UKQCD have $2 - 4\%$ scaling errors for currently measured observables with $a^{-1} = 1$ GeV. [cf. R. D. Mawhinney’s talk today at 17:30]

• Iwasaki gauge action:

\[
S_G = -\frac{\beta}{3} \left[ (1 - 8c_1) \sum_{x, \mu > \nu} P_{\mu\nu}(x) + c_1 \sum_{x, \mu \neq \nu} R_{\mu\nu}(x) \right], \quad c_1 = -0.331
\]

• Möbius domain wall fermion (MDWF):

\[
S_F(m) = \bar{\psi} \left[ \frac{D_{\text{MDWF}}(m)}{D_{\text{MDWF}}(1)} \right] \psi, \quad H = \frac{(b + c)\gamma_5 D_w}{2 + (b - c)D_w}
\]

• dislocation suppressed determinant ratio (DSDR):

\[
det \left[ \frac{H^2(-M_5) + \epsilon_f^2}{H^2(-M_5) + \epsilon_b^2} \right], \quad H(-M_5) = \gamma_5 D_w(-M_5)
\]
• We have produced ensembles with $a^{-1} = 1$ and 2 GeV with closely matched masses.

<table>
<thead>
<tr>
<th></th>
<th>coarse, $S_c$</th>
<th>fine, $S_f$</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>size</td>
<td>$12^3 \times 32 \times 12$</td>
<td>$24^3 \times 64 \times 12$</td>
<td>−</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.633</td>
<td>1.943</td>
<td>−</td>
</tr>
<tr>
<td>$am_l$</td>
<td>0.008521</td>
<td>0.000787</td>
<td>−</td>
</tr>
<tr>
<td>$am_h$</td>
<td>0.065073</td>
<td>0.019896</td>
<td>−</td>
</tr>
<tr>
<td>$a^{-1}$[GeV]</td>
<td>1.015(16)</td>
<td>2.001(18)</td>
<td>−</td>
</tr>
<tr>
<td>$am_{res}(m_l)$</td>
<td>0.007439(86)</td>
<td>0.004522(12)</td>
<td>−</td>
</tr>
<tr>
<td>$m_\pi$[MeV]</td>
<td>307(5)</td>
<td>300(3)</td>
<td>2.3</td>
</tr>
<tr>
<td>$m_K$[MeV]</td>
<td>506(8)</td>
<td>491(5)</td>
<td>3.0</td>
</tr>
<tr>
<td>$m_\Omega$[MeV]</td>
<td>1652(27)</td>
<td>1557(71)</td>
<td>5.9</td>
</tr>
<tr>
<td>$f_\pi$[MeV]</td>
<td>147(2)</td>
<td>138(2)</td>
<td>6.3</td>
</tr>
<tr>
<td>$f_K$[MeV]</td>
<td>166(3)</td>
<td>155(2)</td>
<td>6.8</td>
</tr>
</tbody>
</table>

tuning error

scaling error
Good $O(a^2)$ scaling indicates an approximate RG trajectory in the hyperplane of actions consisting of only ID+MDWF terms.

space of all possible actions $S[U]$
We consider blocking kernels with the form:

\[
G[U_c, U_f] = \prod_{x, \mu} \delta(U_c(x, \mu) - g_b[U_f; x, \mu])
\]

A *naive* blocking kernel:

\[
g_b[U_f] = U_{f,1}U_{f,2}
\]

A single step APE-like blocking kernel:

\[
C = \sum_{U_{f,3}, U_{f,6}} \rightarrow \quad g_b[U_f] = \mathcal{P}[(1 - \alpha)U_{f,1}U_{f,2} + \alpha C/6]
\]
Demon Algorithm*: \( S = \beta_P P + \beta_R R + \cdots \)

- Given a configuration generated by gauge action we can introduce a series of *demon* variables to probe the underlying \( \beta \)'s in the action. \( S_i \) can be plaquette(\( P \)), rectangular(\( R \)), chair loop(\( C \)), twist loop(\( T \)), \( \cdots \)

\[
\int [D U] \int \prod_i [d E_i] \exp \left[ - \sum_i (\beta_i S_i[U] + \beta_i E_i) \right].
\]

- The update scheme consists of two parts,
  1. update \( U \) only.
  2. update \( U \) and \( E_i \)'s at the same time while keeping \( S_i + E_i \) constant. In this case the accept/reject part does not require knowledge of \( \beta_i \)'s.

- Since the \( E_i \)'s integration factorizes we can measure the average value of them and probe the underlying \( \beta_i \)'s.

\[
\langle E_i \rangle = \frac{1}{\beta_i} - \frac{E_{\max}}{\tanh(\beta_i E_{\max})}.
\]

*See [T. Takaishi, 1995].
Determining the blocked coarse action

- Generating a sequence of fine lattices and then blocking them, gives coarse, blocked lattices that appear with a probability proportional to $e^{-S^b_c[U_c]} \propto \int [dU_f] e^{-S_f[U_f]} G[U_c, U_f] / \int [dU_f] e^{-S_f[U_f]}$

  space of all possible actions $S[U]$

- Compare

  hyperplane of actions consist of only ID+MDWF terms $S[U; \beta, m_l, m_h]$
• Apply demon algorithm on the blocked coarse lattice with different $\alpha$.

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\beta_P$</th>
<th>$\beta_R$</th>
<th>$\beta_C$</th>
<th>$\beta_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ID+MDWF</td>
<td>2.035(36)</td>
<td>$-0.1018(33)$</td>
<td>$-0.0026(30)$</td>
<td>$-0.0006(30)$</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>0.617(11)</td>
<td>0.0491(33)</td>
<td>0.0032(32)</td>
<td>0.0010(32)</td>
</tr>
<tr>
<td>$\alpha = 0.5$</td>
<td>1.478(35)</td>
<td>$-0.0020(44)$</td>
<td>0.0043(42)</td>
<td>$-0.0016(43)$</td>
</tr>
<tr>
<td>$\alpha = 0.688$</td>
<td>2.030(28)</td>
<td>$-0.1522(30)$</td>
<td>$-0.0021(24)$</td>
<td>0.0038(24)</td>
</tr>
<tr>
<td>$\alpha = 0.7$</td>
<td>2.069(33)</td>
<td>$-0.1589(33)$</td>
<td>0.0009(27)</td>
<td>$-0.0003(27)$</td>
</tr>
</tbody>
</table>

• We will use $\alpha = 0.688$. 
<table>
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<tr>
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<th>blocked coarse$^\dagger$, $\langle O \rangle^b$</th>
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<td>$a^{-1}$[GeV]</td>
<td>1.015(16)</td>
<td>1.010(16)</td>
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<td>0.007439(86)</td>
<td>0.00847(21)</td>
<td>−</td>
</tr>
<tr>
<td>$am_\pi$</td>
<td>0.3026(13)</td>
<td>0.3144(34)</td>
<td>3.9</td>
</tr>
<tr>
<td>$am_K$</td>
<td>0.4982(11)</td>
<td>0.5072(24)</td>
<td>1.8</td>
</tr>
<tr>
<td>$am_\Omega$</td>
<td>1.628(10)</td>
<td>1.658(13)</td>
<td>1.8</td>
</tr>
<tr>
<td>$af_\pi$</td>
<td>0.14472(64)</td>
<td>0.14999(99)</td>
<td>3.6</td>
</tr>
<tr>
<td>$af_K$</td>
<td>0.16333(47)</td>
<td>0.16791(85)</td>
<td>2.8</td>
</tr>
</tbody>
</table>

$^\dagger$ Valence quark masses are taken to be the same as the coarse ensemble.
\[ \langle t^2 E \rangle \]

The graph shows the evolution of \( \langle t^2 E \rangle \) with respect to time \( t \) for different coarse approximations. The graph compares three curves:

- **blocked coarse** (magenta line)
- **coarse** (green line)
- **coarse** with a dashed line for reference.

The x-axis represents time \( t \), ranging from 0 to 1.4, while the y-axis represents the value of \( \langle t^2 E \rangle \), ranging from 0 to 0.7. The graph illustrates how the energy \( E \) evolves over time for different approximations.
\[
\Lambda_V = \frac{Z_q}{Z_V}
\]

blocked coarse: vector \(\langle \Lambda_V \rangle^b\)

coarse: vector \(\langle \Lambda_V \rangle\)

‡All valence quark masses are taken to be the sea quark masses of the coarse ensemble. Non-exceptional momenta are used. Refer to [R. Arthur, 2011, arXiv:1006.0422].
\[ \Lambda_A = \frac{Z_q}{Z_A} \approx 1 \text{ GeV} \]

```
  blocked coarse : axial vector \langle \Lambda_A \rangle^b
  coarse : axial vector \langle \Lambda_A \rangle
```

![Graph showing \( \Lambda_A = \frac{Z_q}{Z_A} \) vs. \( a^{-1} \)](image)
\[ \Lambda_S = \frac{Z_q}{Z_S} \]

blocked coarse : scalar \( \langle \Lambda_S \rangle^b \)

coarse : scalar \( \langle \Lambda_S \rangle \)
\[ \Lambda_P = \frac{Z_Q}{Z_P} \]

\[ a_P, a^{-1} \sim 1 \text{ GeV} \]
Conclusion

• We have produced $12^3$ and $24^3$ ensembles with 300 MeV pions using the ID+MDWF action.

• Using the demon algorithm and a simple APE-like blocking kernel, we have tuned the single blocking parameter to match the blocked coarse lattice and original ID+MDWF coarse lattice.

• It turns out the blocked coarse action produced by this simple APE-like blocking kernel is only a few percent different from the ID+MDWF coarse action.

So what?

• It might be possible to use this close matching to generate fine lattices from coarse lattices.

• An exact correction factor is likely needed to go from few percent discrepancies to an exact algorithm.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>NLO(370 MeV)</th>
<th>NLO(450 MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_f^I\pi$ [GeV$^2$]</td>
<td>0.059(47)</td>
<td>−0.028(51)</td>
</tr>
<tr>
<td>$c_f^{ID}\pi$ [GeV$^2$]</td>
<td>−0.013(17)</td>
<td>−0.058(19)</td>
</tr>
<tr>
<td>$c_f^I\pi$ [GeV$^2$]</td>
<td>0.049(39)</td>
<td>−0.035(38)</td>
</tr>
<tr>
<td>$c_f^{ID}\pi$ [GeV$^2$]</td>
<td>−0.005(15)</td>
<td>−0.044(14)</td>
</tr>
<tr>
<td>$c_f^K$ [GeV$^2$]</td>
<td>0.081(48)</td>
<td>0.065(45)</td>
</tr>
<tr>
<td>$c_f^{ID}K$ [GeV$^2$]</td>
<td>0.013(15)</td>
<td>0.012(16)</td>
</tr>
<tr>
<td>$c_f^K$ [GeV$^2$]</td>
<td>0.070(41)</td>
<td>0.069(36)</td>
</tr>
<tr>
<td>$c_f^{ID}K$ [GeV$^2$]</td>
<td>0.011(15)</td>
<td>0.019(15)</td>
</tr>
</tbody>
</table>

\[ \langle O \rangle \leftrightarrow \langle O \rangle^b = \frac{1}{Z} \int [dU_c] e^{-S^b_c[U_c]} \mathcal{O}[U_c] \]
\[ = \frac{1}{Z} \int [dU_c] \int [dU_f] G[U_c, U_f] e^{-S_f[U_f]} \mathcal{O}[U_c] \]
\[ = \frac{1}{Z} \int [dU_f] e^{-S_f[U_f]} \int [dU_c] G[U_c, U_f] \mathcal{O}[U_c] \]
\[ = \frac{1}{Z} \int [dU_f] e^{-S_f[U_f]} \tilde{\mathcal{O}}[U_f] \]