Lattice2017

Double-winding Wilson loop in $SU(N)$ Yang-Mills theory

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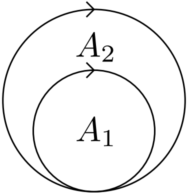
Introduction

Recently, it is argued that double-winding Wilson loops can be used to test the candidates of the confinement mechanism in $SU(2)$ gauge theory in Greensite and Höllwieser, Phys. Rev. D91, 054509 (2015).
In the paper, the following two types of models are considered:

- In the center vortex model, the expectation value of a double-winding Wilson loop is expected to decrease exponentially with the difference of areas $A_2 - A_1$.
- In models associated with the Abelian monopoles, the expectation value is expected to decrease exponentially with the sum of areas $A_1 + A_2$.

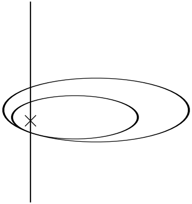
In lattice simulations, the expectation value of a double-winding Wilson loop decreases exponentially with the difference of areas $A_2 - A_1$ in $SU(2)$ Yang-Mills theory.



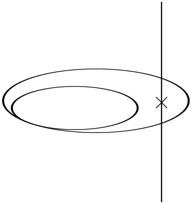
Difference-of-areas behavior in center vortex model

We assume that the loop is so large that we can neglect the thickness of center vortices. Generally, if a vortex links the contour once, the phase factor is multiplied by -1, which is an element of the center of $SU(2)$.

- If a center vortex pierces A_1, then the vortex links the contour twice and hence the phase factor is multiplied by $(-1)^2$, which means there is no effect.
- If a center vortex pierces $A_2 - A_1$, then the vortex links the contour once and hence the phase factor is multiplied by -1.



$U[C] \to (-1)^2 U[C] = U[C]$



$U[C] \to -U[C]$

The situation is the same as if the contour is single winding and has the minimal area $A_2 - A_1$. $\Rightarrow$ difference-of-areas law

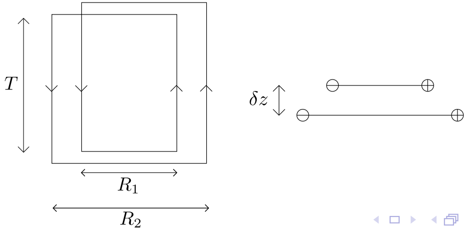
Sum-of-areas behavior in the models associated with Abelian monopole

The contour is rectangular. $\Rightarrow \quad W(C) \to \exp(-TV(R_1, R_2))$

In the dual superconductivity picture the electric flux between quark and antiquark forms a tube due to the condensation of the monopoles.
If two pairs are separated, the total energy is the sum of energy of each flux tube.

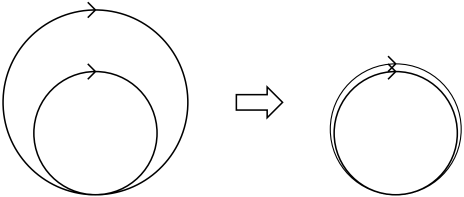
If $\delta z \to 0$
TYPE I: The flux tubes attract each other and therefore the total energy is smaller than the separated case.
TYPE II: The flux tubes repel each other and therefore the total energy is almost sum of the energy of each flux.

In either case, the expectation value of double-winding Wilson loop is expected to decrease exponentially with either A_1 and A_2.



Purpose of this work

Our main purpose is to extend this argument to $SU(N)$ case for $N \geq 3$. To do this we need to know the correct behavior of the expectation value of a double-winding Wilson loop in $SU(N)$ Yang-Mills theory. In this talk, in the case that the two loops are identical we derive the behavior of the expectation value of the double-winding Wilson loop by using group identities and assuming the Casimir scaling irrespective of the dimensionality.



Outline

1. Double-winding Wilson loop with identical loops
2. N-times-winding Wilson loop
3. Double-winding Wilson loop with non-identical loops

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Double-winding Wilson loop with identical loops

In $SU(2)$ case, the expectation value of a double-winding Wilson loop with identical loops can be estimated by using the group identity

$$\mathrm{tr}\, U^2 = \mathrm{tr}\, U_A - 1,$$

where U is an arbitrary element of the group and U_A is the image of U under the adjoint representation.

Similarly, in $SU(N)$ case, we can also estimate it by using the group identity

$$\mathrm{tr}\, U^2 = \mathrm{tr}\, U_{\square\square} - \mathrm{tr}\, U_{\substack{\square \\ \square}},$$

where the Young diagrams denote the representations.

Using this group identity the double-winding Wilson loop with identical loops can be written as

$$W_2 = \frac{N+1}{2} W_{\square\square} - \frac{N-1}{2} W_{\substack{\square \\ \square}}.$$

The expectation value

By using this relation and assuming Casimir scaling, we can estimate the expectation value for the intermediate size contour.

The ratios of the values of the quadratic Casimir in the representations and that in the fundamental representation are

$$C_2\left(\square\square\right)/C_2(F) = \frac{2(N+2)}{N+1},$$

$$C_2\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)/C_2(F) = \frac{2(N-2)}{N-1}.$$

By assuming the Casimir scaling, therefore, in the case that the minimal area S is intermediate, we can expect

$$\langle W_2 \rangle \simeq a_N \exp\left(-2\frac{N+2}{N+1}\sigma_F S\right) - b_N \exp\left(-2\frac{N-2}{N-1}\sigma_F S\right), \quad a_N, b_N > 0.$$

- For sufficiently large S the dominant part is the second term.
- For $N \geq 3$, this behavior is consistent with neither difference-of-areas law nor sum-of-areas law.

- In $N = 3$ case, the behavior of the double-winding Wilson loop with identical loops is same as the single-winding one:

$$\langle W_2 \rangle \sim -b_3 e^{-\sigma_F S}.$$

- In large N limit, it is consistent with the sum-of-areas law:

$$\langle W_2 \rangle \sim -b_\infty e^{-2\sigma_F S}.$$

In the case that N is large, because the "string tension" approaches $2\sigma_F$, we need accurate calculation in order to distinguish the correct behavior from the behavior in the model associated with Abelian monopoles.

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2. N-times-winding Wilson loop
3. Double-winding Wilson loop with non-identical loops

N-times-winding Wilson loop

We can more easily distinguish the correct behavior from the behavior in models associated with Abelian monopoles by using N-times-winding Wilson loops. To see this, we consider the general m-times-winding Wilson loop with identical loops. In this case, we can also estimate the behavior of the expectation value by using the group identity.

General m-times-winding Wilson loop

The m-times-winding Wilson loop is defined as the Wilson loop for a contour which consists of m identical loops.
The trace of the mth power of a group element U can be written using the trace in the higher dimensional representation:

for $m < N$

$$\mathrm{tr}\, U^m = U_{\Box\Box\cdots\Box} - U_{\begin{array}{l}\Box\Box\cdots\Box\\ \Box\end{array}} + \cdots + (-1)^{l-1} U_{\begin{array}{l}\Box\Box\cdots\Box\\ \Box\\ \vdots\\ \Box\end{array}} + \cdots + (-1)^{m-1} U_{\begin{array}{l}\Box\\ \Box\\ \vdots\\ \Box\end{array}},$$

where all diagrams are "L-shape" or "I-shape", there are m boxes in each diagram, and there are l raws in the diagram in lth term,
and for $m \geq N$

$$\mathrm{tr}\, U^m = U_{\Box\Box\cdots\Box} - U_{\begin{array}{l}\Box\Box\cdots\Box\\ \Box\end{array}} + \cdots + (-1)^{l-1} U_{\begin{array}{l}\Box\Box\cdots\Box\\ \Box\\ \vdots\\ \Box\end{array}} + \cdots + (-1)^{N-1} U_{\begin{array}{l}\Box\Box\cdots\Box\\ \Box\\ \vdots\\ \Box\end{array}},$$

where there are N terms.

Using this formula and assuming the Casimir scaling, in the case that the loop has an intermediate size, we can expect

$$\langle W_m \rangle \simeq \begin{cases} (-1)^{m-1} a_{Nm} \exp\left(-\frac{m(N-m)}{N-1}\sigma_F S\right) & \text{for } m < N, \\ (-1)^{N-1} a_{NN} & \text{for } m = N, \\ (-1)^{m-1} a_{Nm} \exp\left(-\frac{m(m-N)}{N+1}\sigma_F S\right) & \text{for } m > N. \end{cases}$$

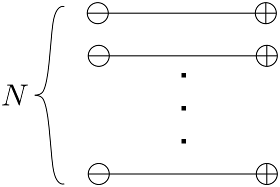
The reason why, for $m = N$, the expectation value does not decrease exponentially is as follows. The young diagram

corresponds to the trivial representation if the number of the boxes is N. Hence, if $m = N$, the last term is equal to $(-1)^{N-1}$. In this way we see that $\langle W_N \rangle$ does not follow the area law.

It seems that we can use W_N similarly to the double winding Wilson loop in $SU(2)$ case.

The behavior of the expectation value of N-times-winding Wilson loop in the center vortex model and in models associated with Abelian monopoles

- If a center vortex pierces the minimal area of a N-times-winding Wilson loop, there is no effect because the Nth power of an arbitrary element of the center is one. The expectation value is expected not to decrease exponentially with the area.
- Naively, in the models associated with monopole, the expectation value of N-times-winding Wilson loop is expected to decrease exponentially with NS.



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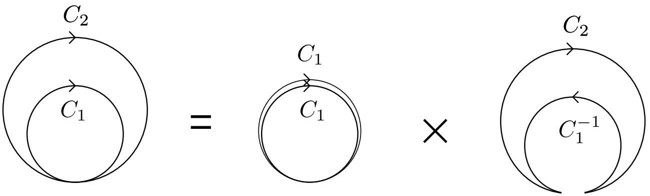
Double-winding Wilson loop with non-identical loops

I would like to conjecture the behavior of the double-winding Wilson loop average in the case that two loops are non-identical. To do this, we assume factorization for non-intersecting coplanar loops.

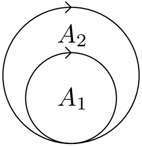
$$\langle \operatorname{tr} U(C) \rangle = \langle \operatorname{tr} U^2(C_1) U^{-1}(C_1) U(C_2) \rangle \simeq \operatorname{tr} \langle U^2(C_1) \rangle \langle U^{-1}(C_1) U(C_2) \rangle$$

This assumption is consistent with usual area law.



By substituting the result in the identical loops case and usual single winding case, we obtian the dominant part as

$$W_2 \sim -b_N \exp\left(-2\frac{N-2}{N-1}\sigma_F A_1 - \sigma_F(A_2 - A_1)\right)$$
$$= -b_N \exp\left(-\sigma_F\left(\frac{N-3}{N-1}A_1 + A_2\right)\right),$$



which is consistent with $A_1 = A_2$ case and $A_1 = 0$ case.

According to Bralic (1980), in two-dimensional spacetime, we can calculate double-winding Wilson loops exactly as

$$W_2 = \frac{N+1}{2}\exp\left(-\frac{g^2}{2}\frac{N^2-1}{2N}\left(\frac{N+3}{N+1}A_1 + A_2\right)\right)$$
$$-\frac{N-1}{2}\exp\left(-\frac{g^2}{2}\frac{N^2-1}{2N}\left(\frac{N-3}{N-1}A_1 + A_2\right)\right).$$

Therefore at least in two-dimensional spacetime, the above behavior is confirmed.

Summary

- In the case that two loops are identical, we can expect the behavior of a double-winding Wilson loop expectation value by using the group identity and assuming the Casimir scaling in $SU(N)$ case. It is consistent with neither difference-of-areas law nor sum-of-areas law for $N \geq 3$. In large N limit, it is consistent with sum-of-areas law.
- We can also expect the behavior of a m-times-winding Wilson loop expectation value similarly.
- N-times-winding Wilson loop expectation value in $SU(N)$ gauge theory behaves similaly to the double-winding Wilson loop in $SU(2)$ gauge theoery.
- In the case that two loops are non-identical, we have conjectured the behavior of a double-winding Wilson loop expectaion value, which is consistent with non-identical case. It is confirmed in two-dimensional Yang-Mills theory.

In the next talk, it is shown that a lattice simulation is consistent with the result of this talk for double-winding Wilson loop in $SU(3)$ gauge theory.