



Lattice study of area law for double-winding Wilson loops

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Introduction

□ Wilson loop is a gauge invariant and important operator for the lattice study.

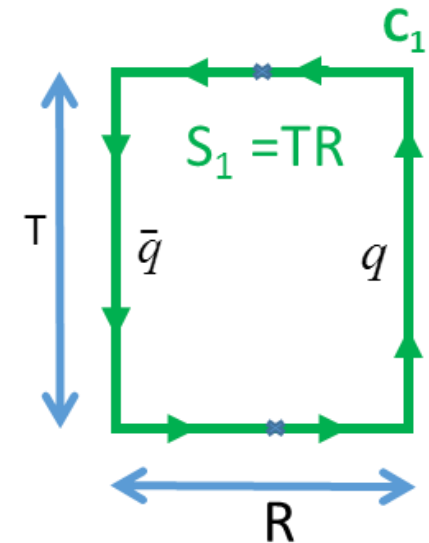
□ A single winding (usual) Wilson loop operator

- Quark and antiquark source of the fundamental representation
- The Wilson loop average gives the static potential between quark and antiquark
- The area law or the linear potential

$$\langle W(C) \rangle \simeq \exp(-\sigma_1 S_1)$$

• For the study of quark confinement

- Non-Abelian Stokes theorem for Wilson loop operator
- Dual super conductivity picture
 - ✓ The restricted field dominance for the string tension
 - ✓ The magnetic monopole dominance for the string tension
 - ✓ Dual Meissner effect (Chromo-electric flux tube)



Introduction (II)

□ Double winding Wilson loop

Wilson loop operator with path $C=(C_1,C_2)$

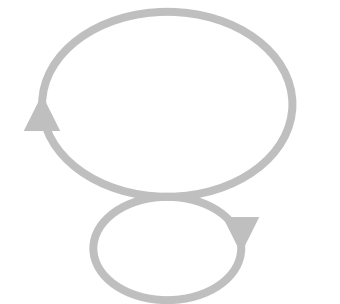
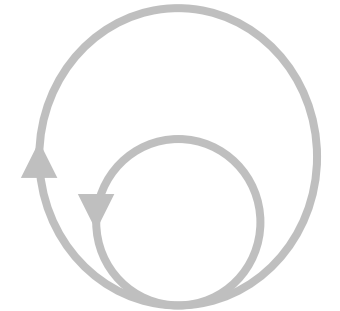
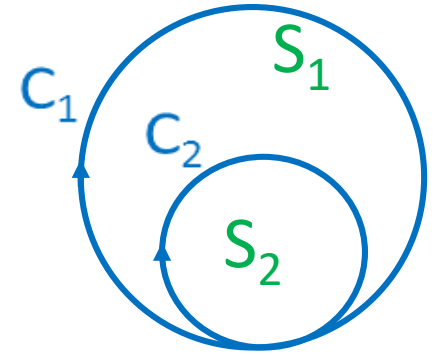
➤ How about string tension for the minimum surface of the Wilson loop ?

- Sum of areas behavior $\langle W(C_1 \times C_2) \rangle \simeq \exp[-\sigma(S_1 + S_2) - \mu P]$
- Difference of area behavior $\langle W(C_1 \times C_2) \rangle \simeq \alpha \exp[-\sigma|S_1 - S_2|]$
- Others

➤ SU(2) case [Greensite et.al [PRD91 054509 \(2015\)](#)]

- Difference of area behavior
- cf U(1) , Abelian sum of areas behavior

How about SU(N) (N>2) case ?



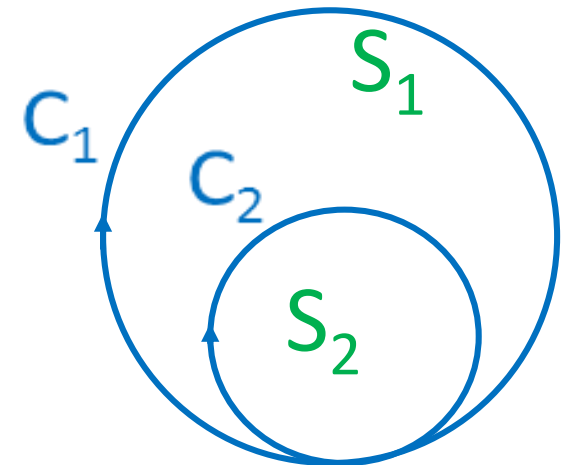
Double Winding Wilson loop

Contours which wind once around a loop C_1 and once around a loop C_2 ,

- ✓ the two co-planar loops share one point in common,
- ✓ C_2 lies entirely in the minimal area of C_1 .
- ✓ C_1 and C_2 are in the same direction.
- ✓ S_1 is the minimum area framed by C_1
- ✓ S_2 is the minimum area framed by C_2

The purpose of this work is to investigate the double-winding Wilson loop of $SU(N)$ Yang-Mills theory and to know the correct behavior of its expectation values such as:

- ◆ Gauge group dependence
- ◆ Relation to the N-arity
- ◆ Relation to the non-Abelian Stokes theorem and the dual superconductivity



CONTENTS

- Introduction
- Study of Double-winding Wilson loop
 - Strong coupling expansion
 - Numerical simulations
- Summary and Discussion

Study of Double-winding Wilson loop

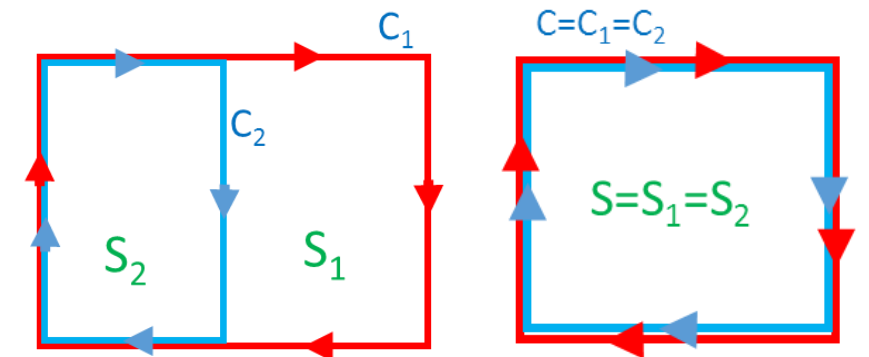
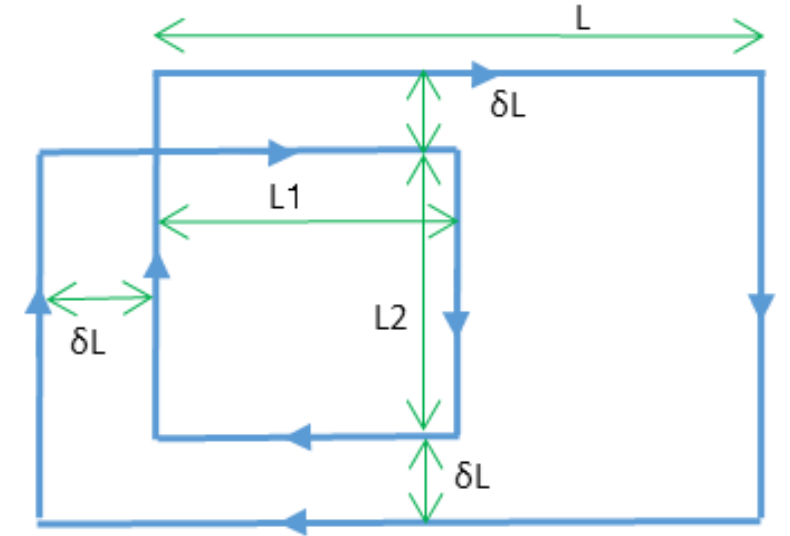
In this study we consider the double-winding Wilson loop as the right figures

□ SU(N) Yang-Mills theory : Wilson action

$$S_g = \beta \sum_{\square} \text{Re tr}(1 - U_{\square}) \quad U_{\square} = U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^{\dagger} U_{x,\nu}^{\dagger}$$

□ discuss the expectation value of such double-winding loops.

- By using the strong coupling expansion
- By using numerical simulation



Strong coupling expansion

Strong coupling expansion

SU(N) group integral

$$\int dU 1 = 1$$

$$\int dU U_{ab} = 0$$

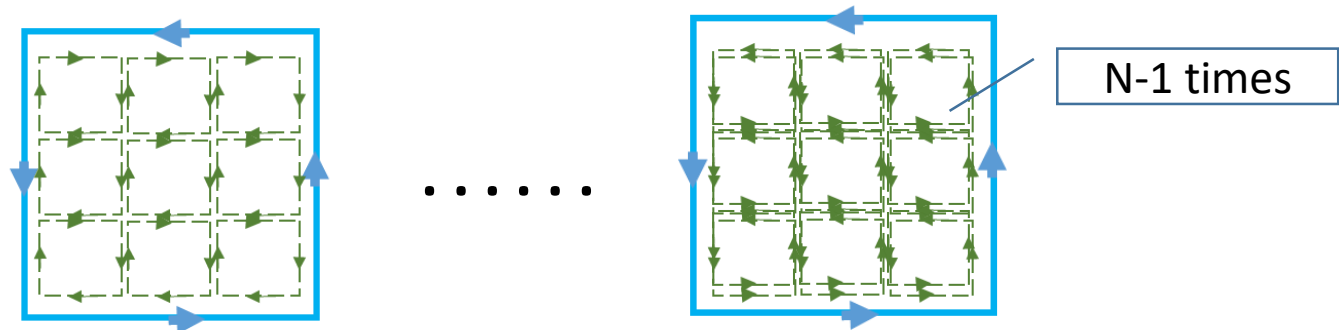
$$\int dU U_{ab} U_{kl}^\dagger = \frac{1}{N} \delta_{al} \delta_{bk}$$

$$\int dU U_{a_1 b_1} U_{a_2 b_2} \cdots U_{a_N b_N} = \frac{1}{N!} \epsilon_{a_1 a_2 \cdots a_N} \epsilon_{b_1 b_2 \cdots b_N}$$

$$\int dU U_{a_1 b_1} U_{a_2 b_2} \cdots U_{a_M b_M} = 0, \quad (M \neq 0 \pmod{N})$$

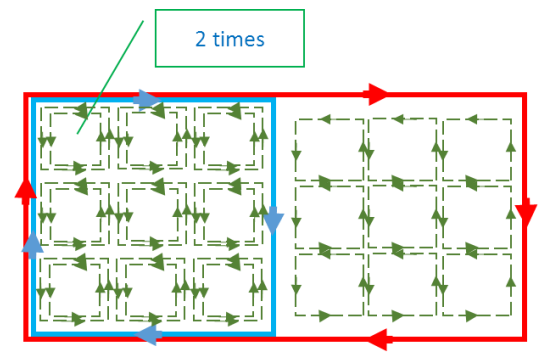
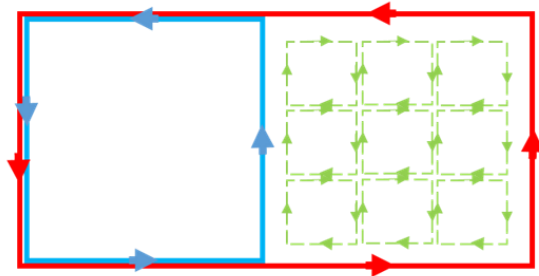
$$\int dU U_{ab} U_{cd} U_{ij}^\dagger U_{kl}^\dagger = \frac{1}{(N^2 - 1)} \left[\delta_{aj} \delta_{bi} \delta_{cl} \delta_{dk} + \delta_{al} \delta_{bk} \delta_{cj} \delta_{di} - \frac{1}{N} (\delta_{aj} \delta_{bk} \delta_{cl} \delta_{di} + \delta_{al} \delta_{bi} \delta_{cj} \delta_{dk}) \right]$$

- Example which contribute to expectation of operators: leading and corrections.



SU(2) case

- Leading term

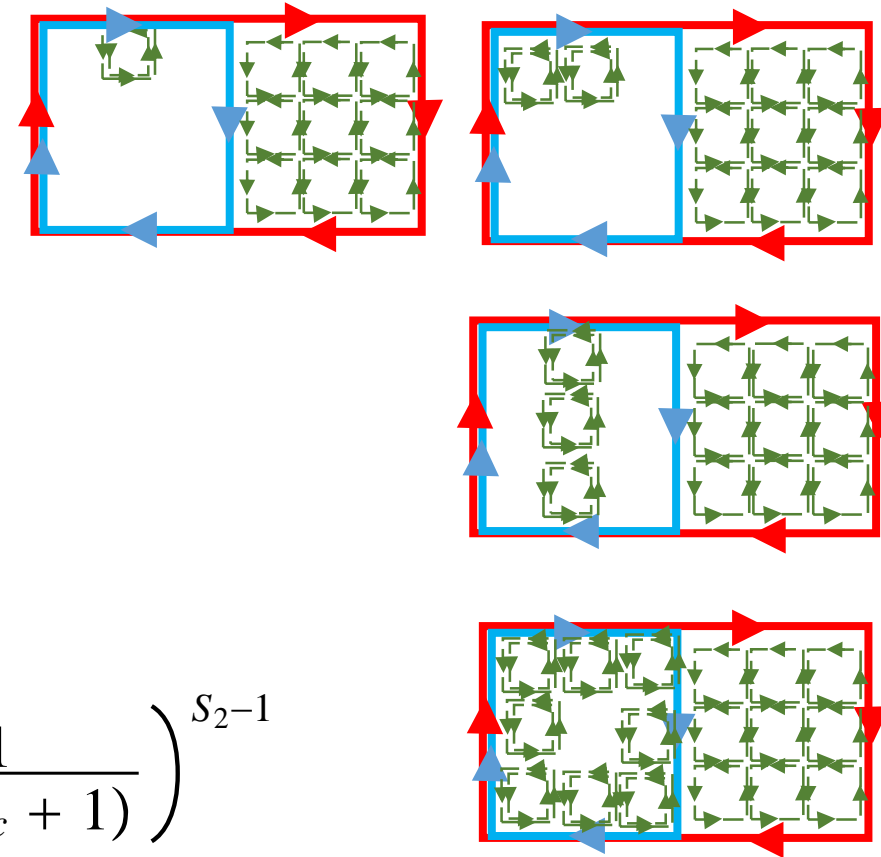


$$\langle W(C) \rangle|_{x_2} = -2 \left(\frac{\beta}{N_c} \right)^{S_1 - S_2}$$

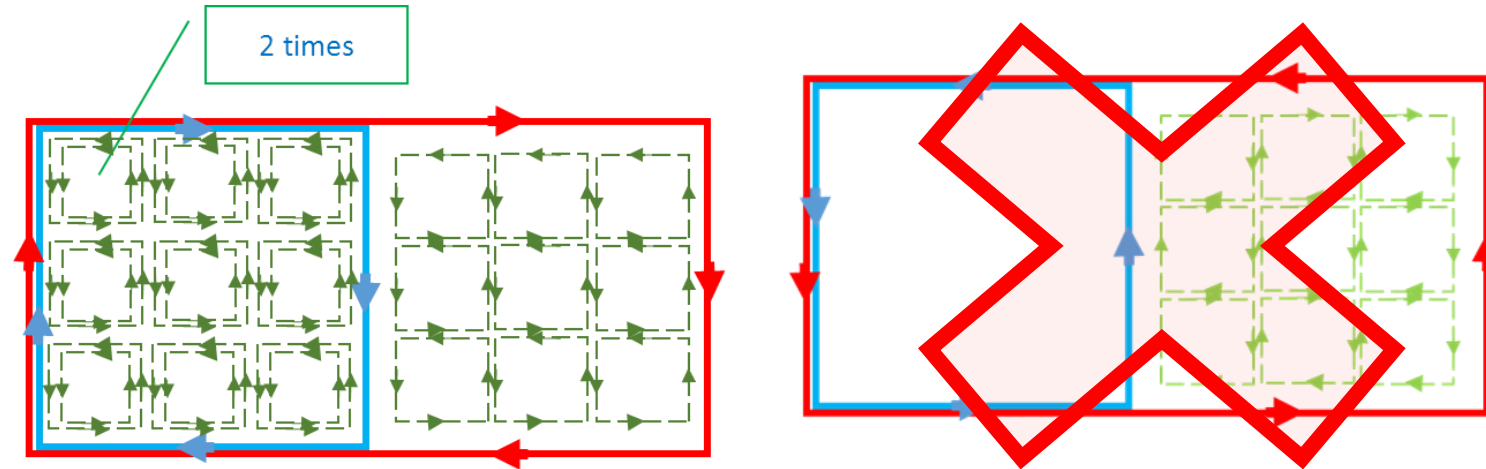
$$\langle W(C) \rangle|_{y_2} = -\frac{1}{2} q \beta^{2S_2} \left(\frac{\beta}{N_c} \right)^{S_1 - S_2}$$

$$q = \left(\frac{1}{N_c(N_c - 1)} \right)^{S_2 - 1} - \left(\frac{1}{N_c(N_c + 1)} \right)^{S_2 - 1}$$

- Correction terms



Abelian, U(1) case



- Sum of areas behavior

$$\langle W(C_1 \times C_2) \rangle = \exp(-\sigma(S_1 + S_2))$$

Group integration

$$\int dg = 1$$

$$\int dgg = 0$$

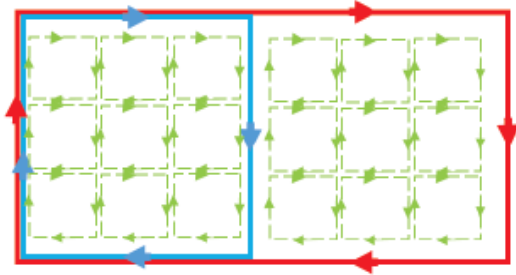
$$\int dggg^\dagger = 1$$

$$\int dggg = 0$$

$$\int dgggg^\dagger g^\dagger = 1$$

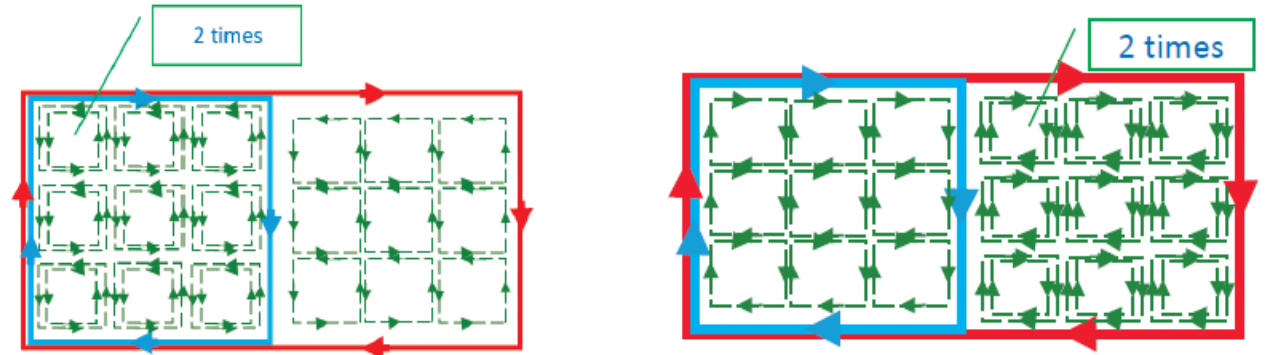
SU(3) case

Leading term



$$\langle W(C) \rangle|_{x_3} = -\left(\frac{\beta}{N_c}\right)^{S_1}$$

Corrections

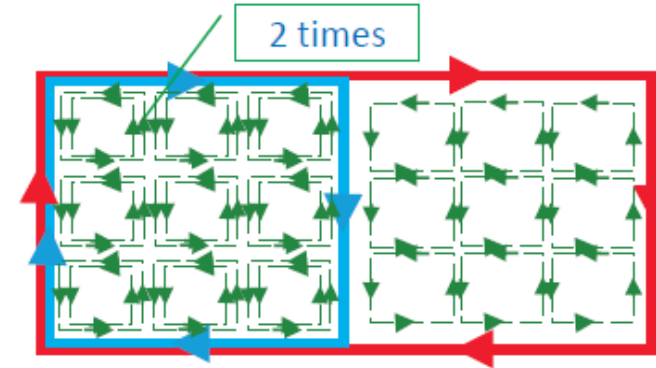
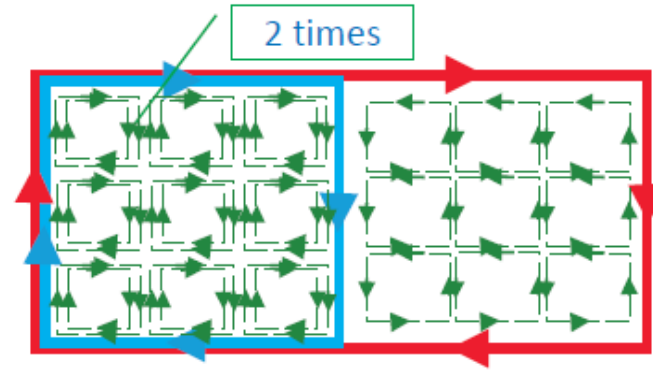


$$\langle W(C) \rangle|_{y_3} = -\frac{1}{2} q(S_2) \beta^{2S_2} \left(\frac{\beta}{N_c}\right)^{S_1 - S_2}$$

$$\langle W(C) \rangle|_{z_3} = -\frac{1}{2} q(S_1 - S_2) \left(\frac{\beta}{N_c}\right)^{S_2} \beta^{2(S_1 - S_2)}$$

$$q(S) = \left(\frac{1}{N_c(N_c-1)}\right)^{S-1} - \left(\frac{1}{N_c(N_c+1)}\right)^{S-1}$$

SU(4)



Both graphs give the equal contribution:

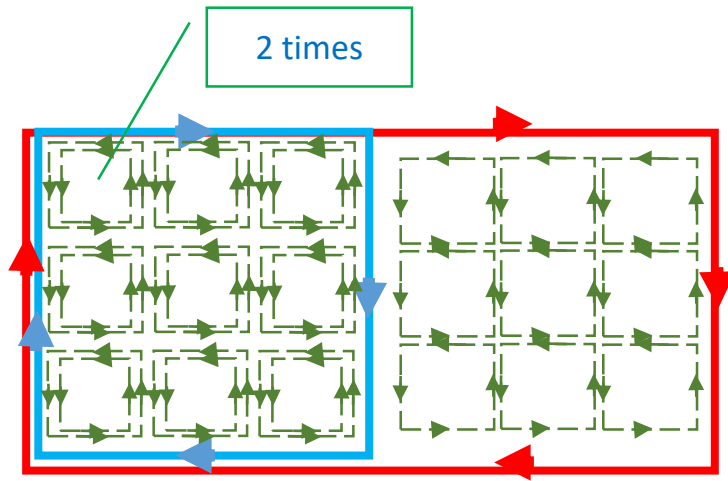
$$\langle W(C) \rangle|_{x_4} = -\frac{1}{2} q(S_2) \beta^{2S_2} \left(\frac{\beta}{N_c} \right)^{S_1 - S_2}$$

$$\langle W(C) \rangle|_{y_4} = -\frac{1}{2} q(S_2) \beta^{2S_2} \left(\frac{\beta}{N_c} \right)^{S_1 - S_2}$$

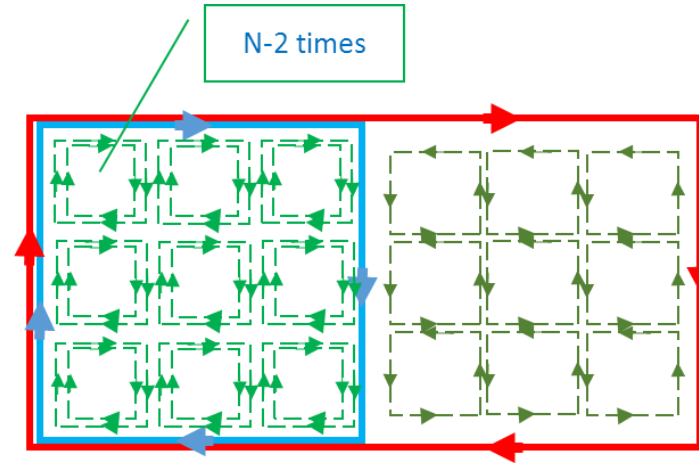
$$q(S) = \left(\frac{1}{N_c(N_c-1)} \right)^{S-1} - \left(\frac{1}{N_c(N_c+1)} \right)^{S-1}$$

SU(N_c) N_c > 4 case

- Leading term



- Correction term



In case of $N_c > 4$, the leading graph is replaced.

$$\langle W(C) \rangle|_{y_{N_c}} = -\frac{1}{2} q(S_2) \beta^{2S_2} \left(\frac{\beta}{N_c} \right)^{S_1 - S_2}$$

$$q \rightarrow \left(\frac{1}{N_c} \right)^{2S_2 - 1} \quad \text{for large } N_c$$

$$q(S) = \left(\frac{1}{N_c(N_c - 1)} \right)^{S-1} - \left(\frac{1}{N_c(N_c + 1)} \right)^{S-1}$$

$$\langle W(C) \rangle|_{y_{N_c}} \rightarrow \left(\frac{\beta}{N_c} \right)^{S_1 + S_2}$$

Summary

- SU(2) case : difference of area low

$$\langle W(C) \rangle \simeq \exp(-\sigma_2(S_1 - S_2))$$

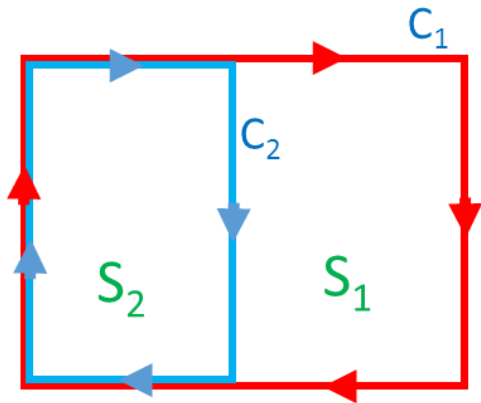
- SU(3) case: neither of difference of areas
lows sum of areas lows

$$\langle W(C) \rangle \simeq \exp(-\sigma_3 S_1)$$

- SU(N) (N>3) case : sum of area low

$$\langle W(C) \rangle = -\frac{1}{2} \exp\{-\sigma'_{N_c}(2S_2) - \sigma_{N_c}(S_1 - S_2)\}$$

$$\sigma_{N_c} \simeq \sigma'_{N_c} \text{ for large } N_c$$



Numerical Simulation

Numerical simulation

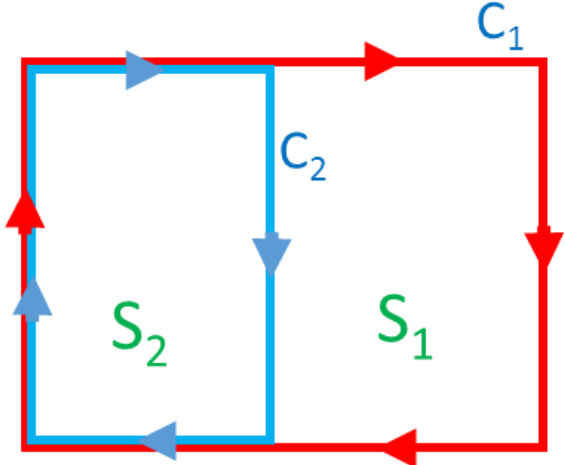
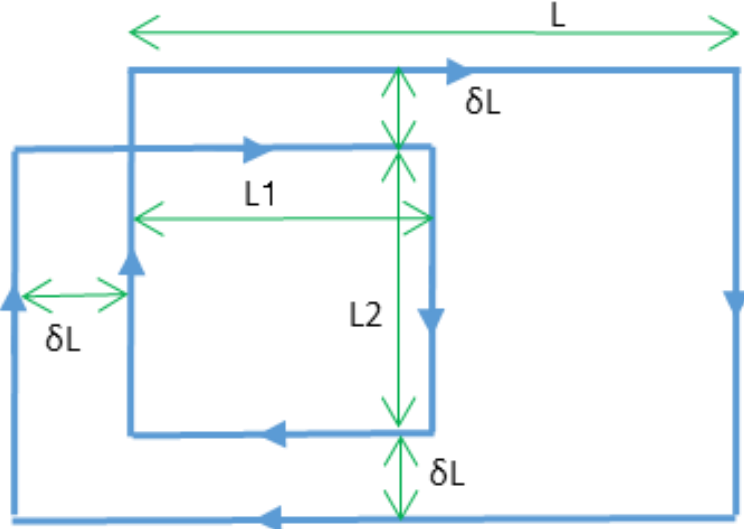
SU(2) case

- Standard Wilson action
- 32^4 lattice $\beta=2.6$ pseudo heat bath
- Gauge links are smeared by using APE smearing method

SU(3) case

- Standard Wilson action
- 24^4 lattice $\beta=6.2$ pseudo heat bath with over relaxation
- Gauge links are smeared by using APE smearing method

SU(2) case ($\delta L=0$)



Double Wilson loop : SU(2) case

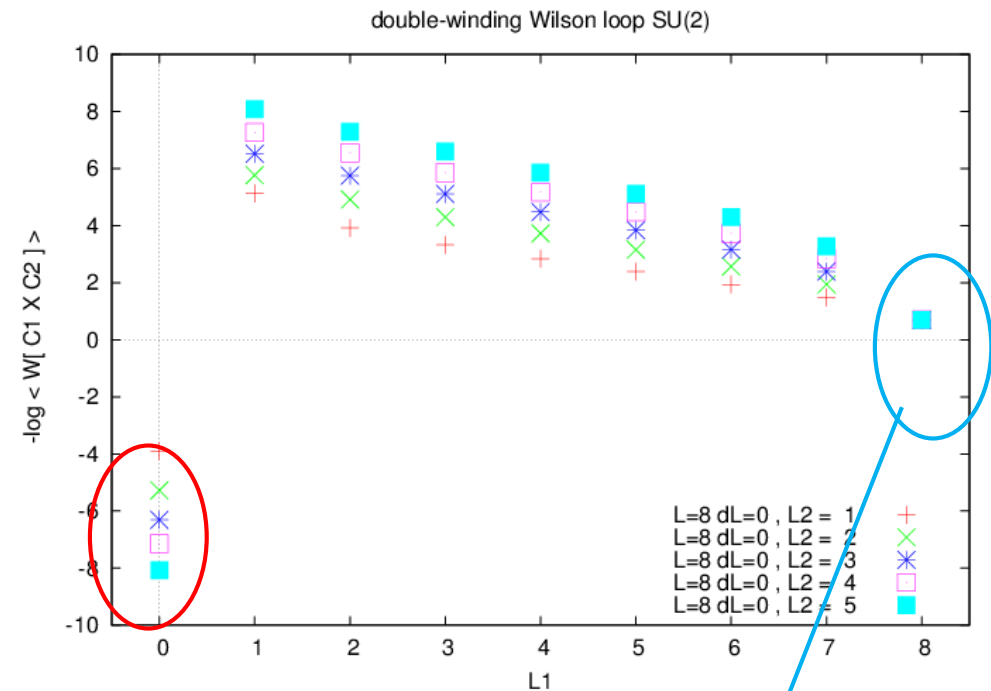
The result by Greensite et.al. [PRD91 054509 \(2015\)](#) is reproduced.

Normal scale L=8 dL=0



L1=0 , single-winding

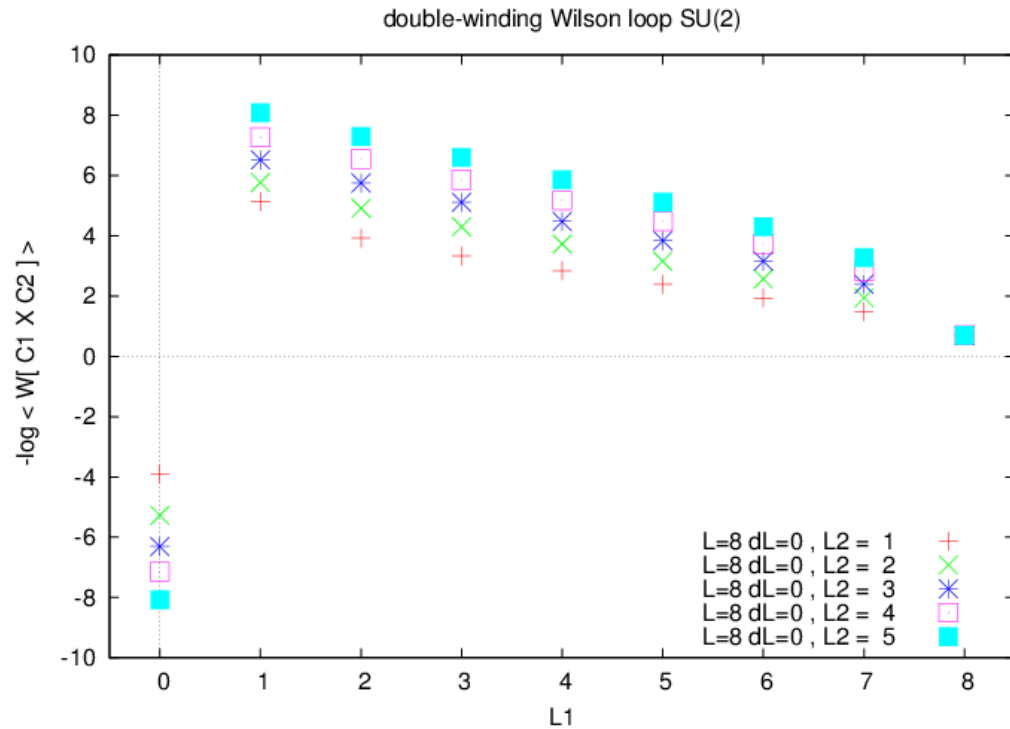
Log scale L=8, dL=0



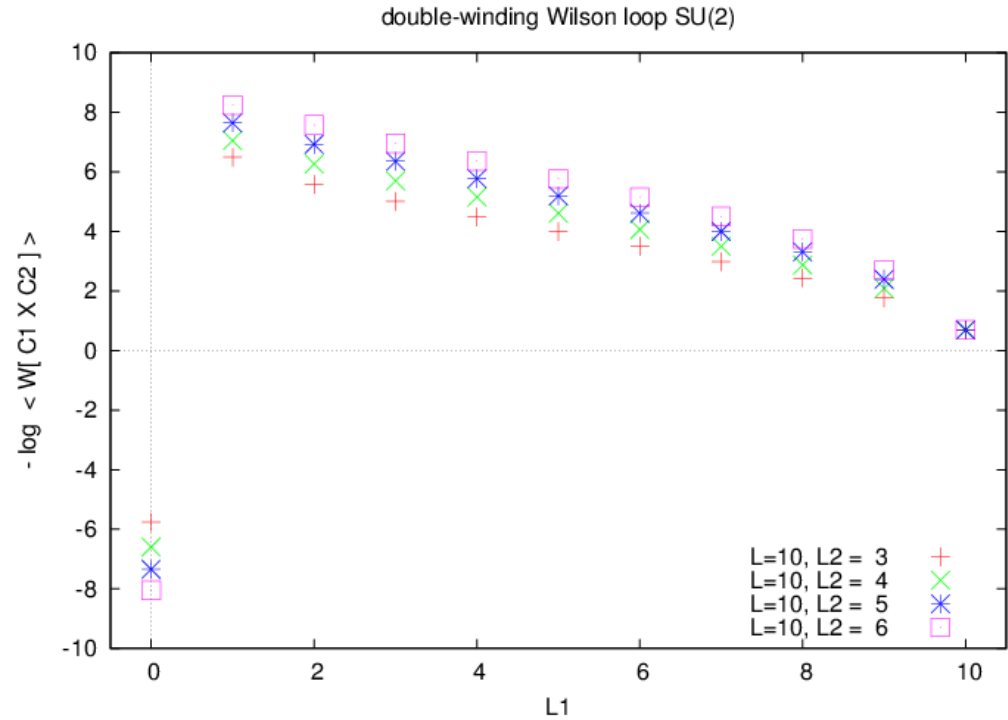
L1=8 double-winding with identical loops

Double Wilson loop : SU(2) case

Log-scale L=8 dL=0



Log-scale L=10, dL=0



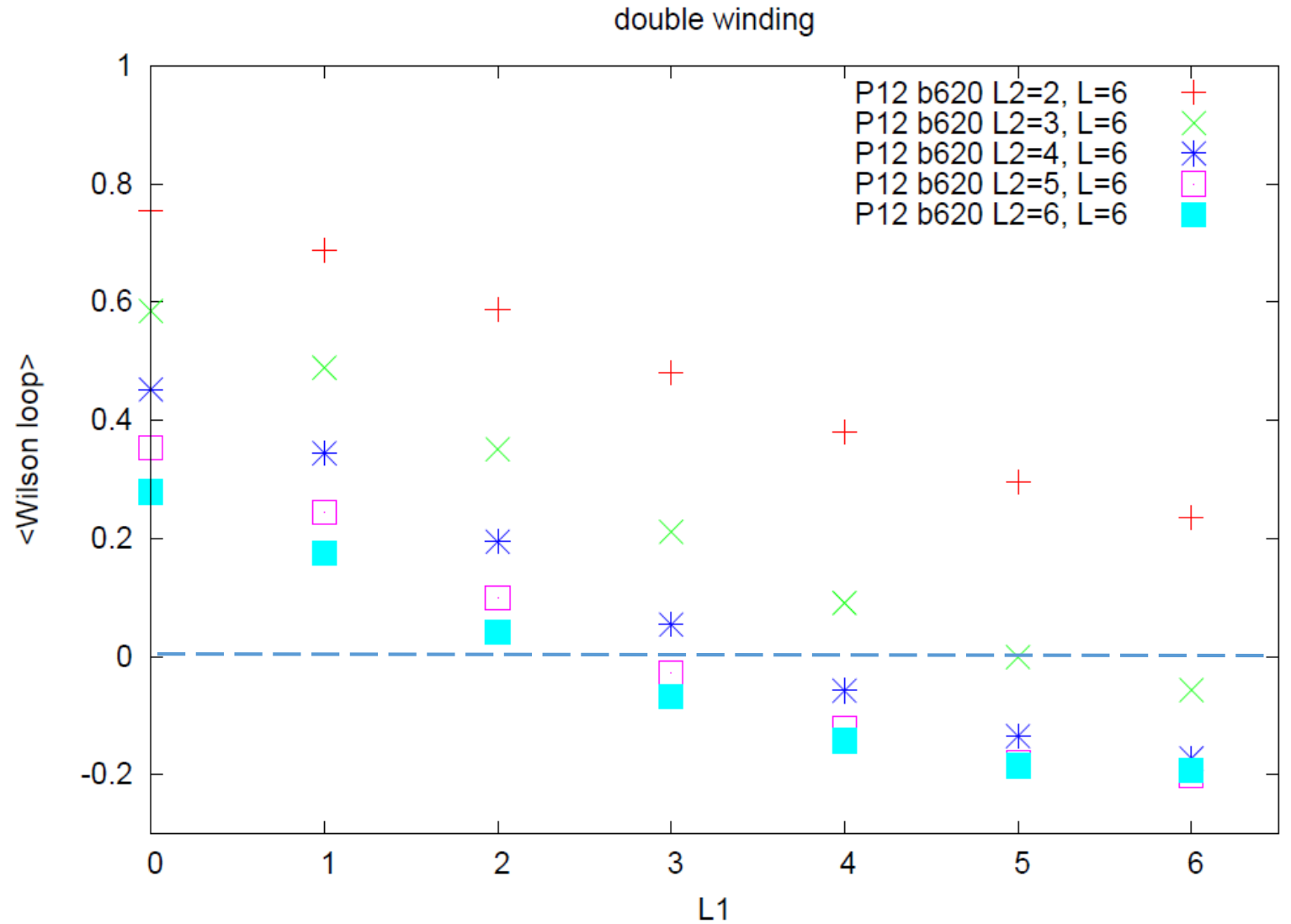
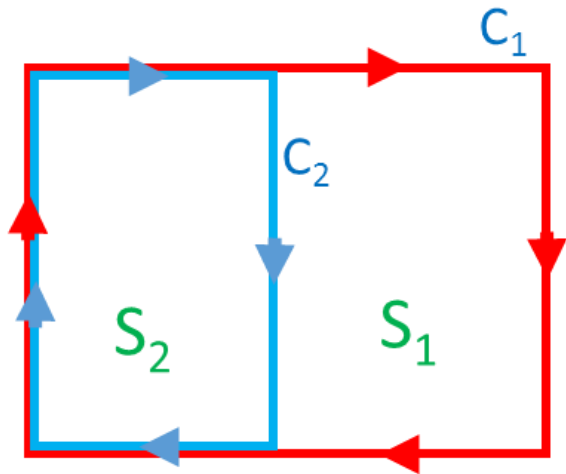
SU(3) case

For $L=6$ the APE smearing with weight = 0.2 are applied 8 times.

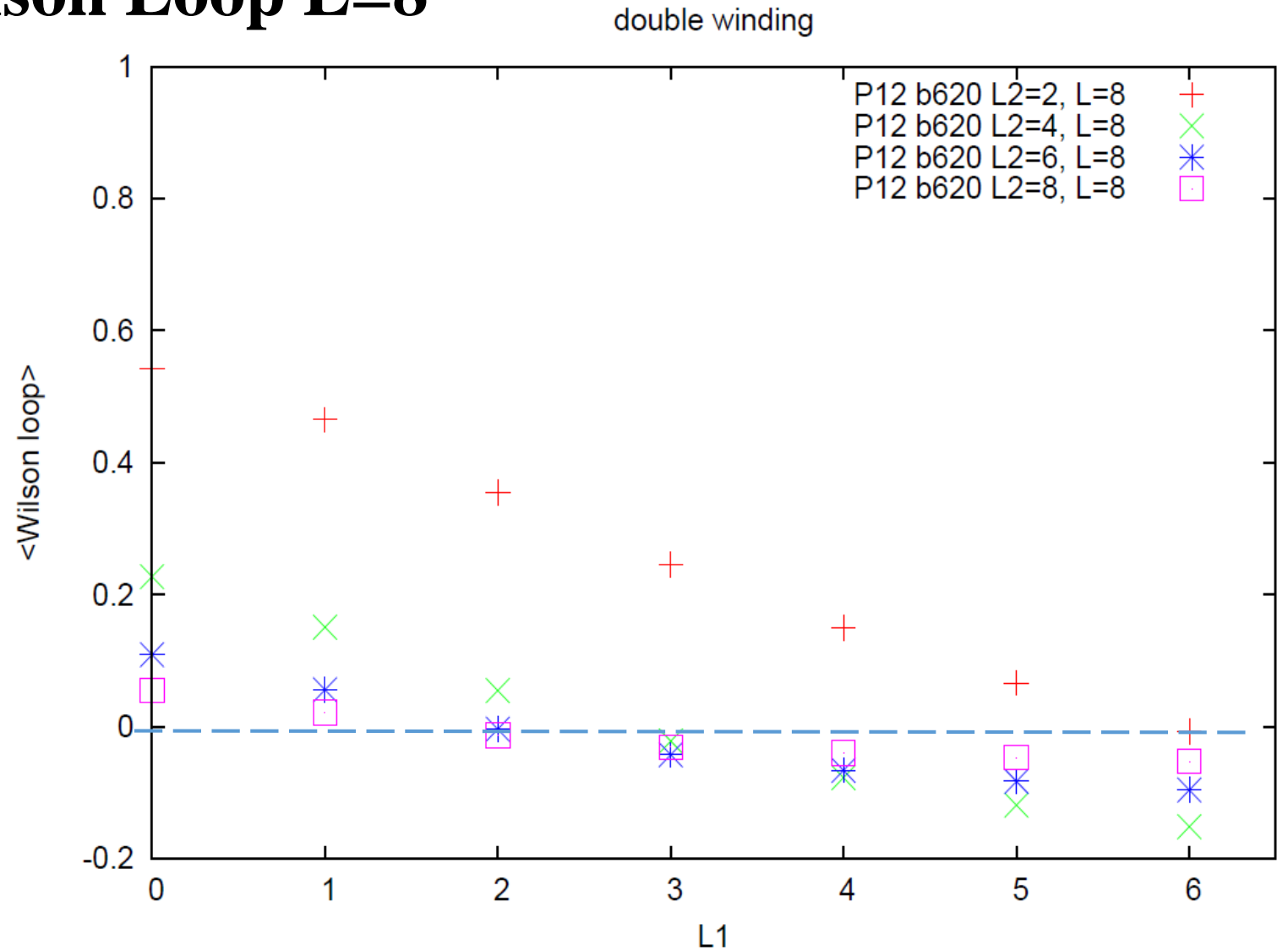
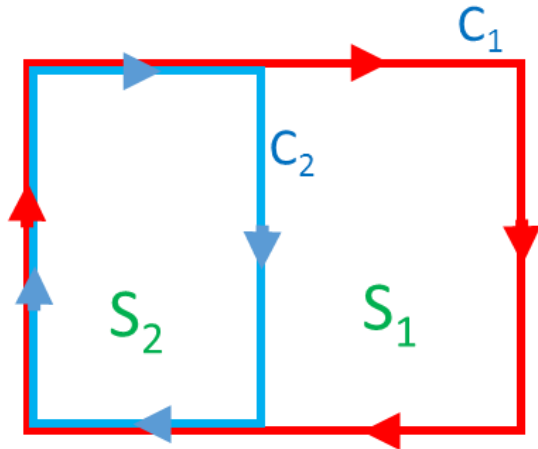
For $L=8$ the APE smearing with weight = 0.1 are applied 8 times.

The numerical results are **preliminary**

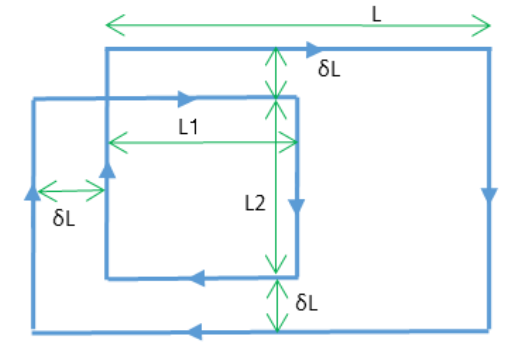
Double-winding Wilson loop $L=6$



Double-winding Wilson Loop $L=8$

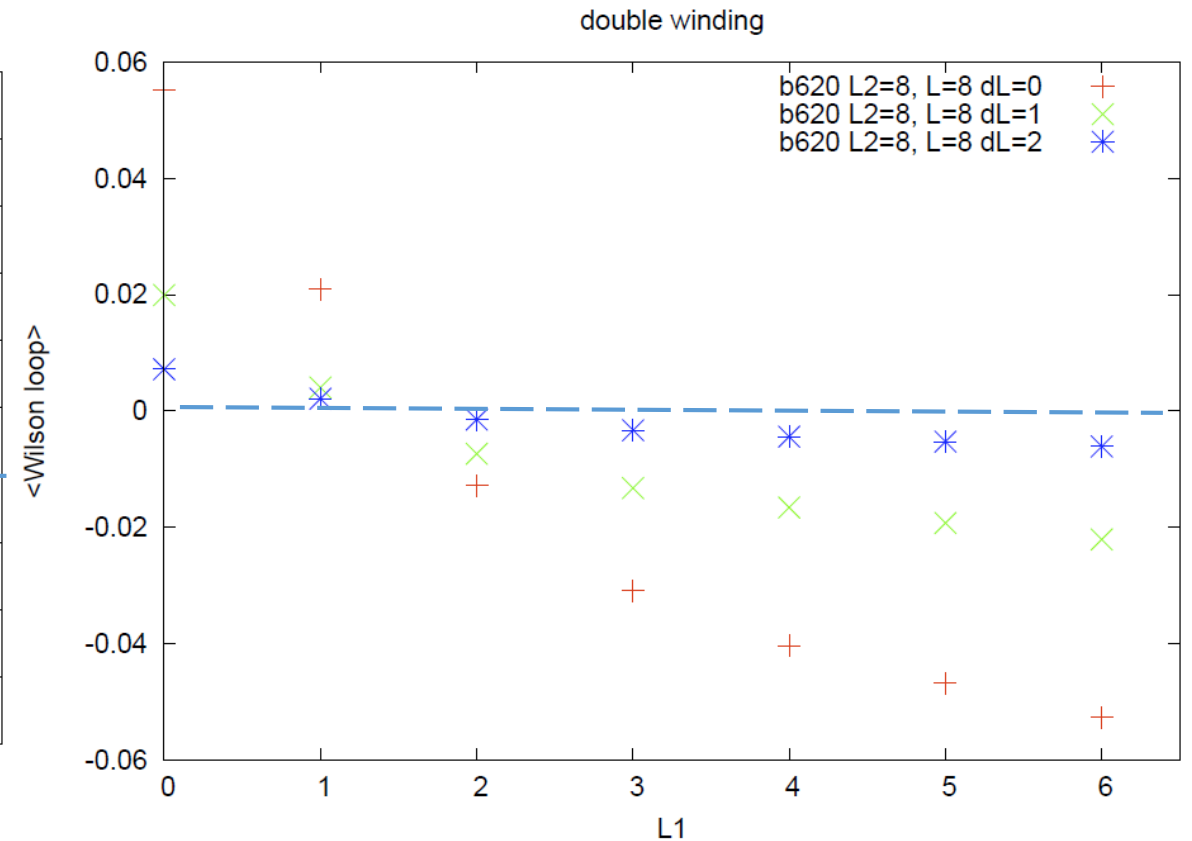
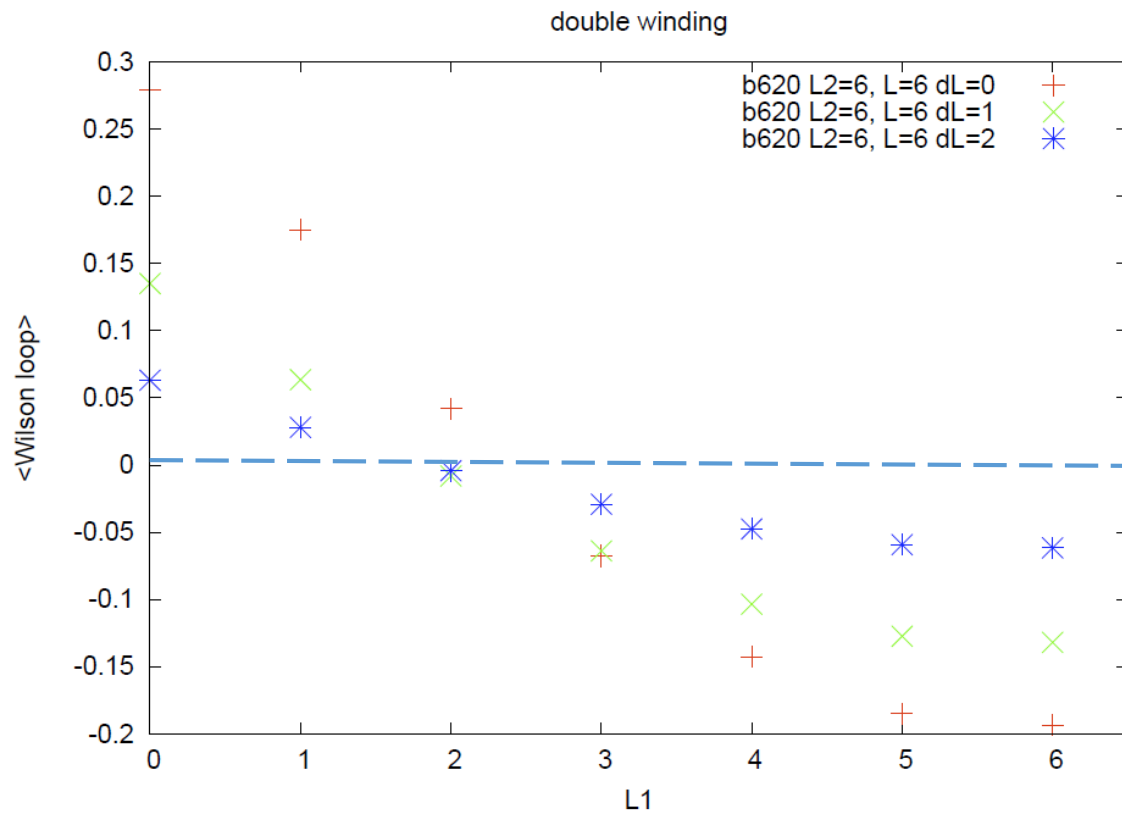


Double Winding Wilson loop (δL dependence)

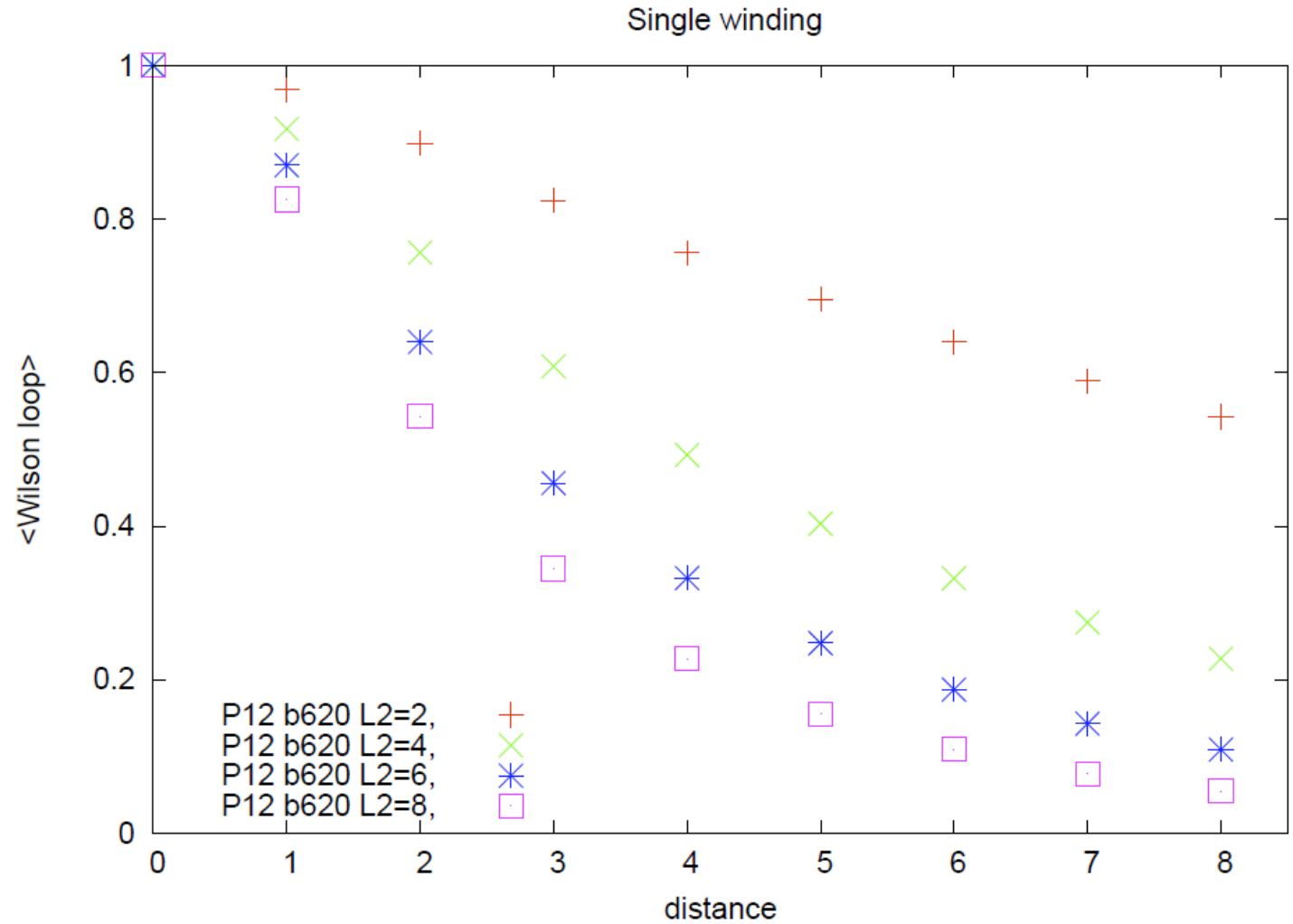
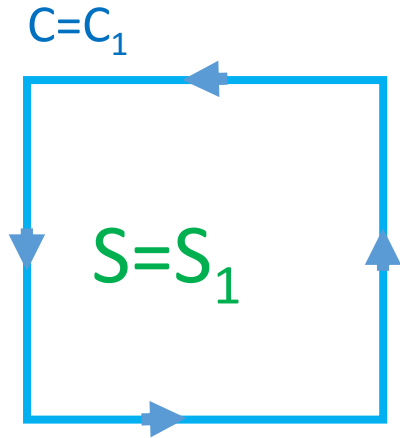


L=6

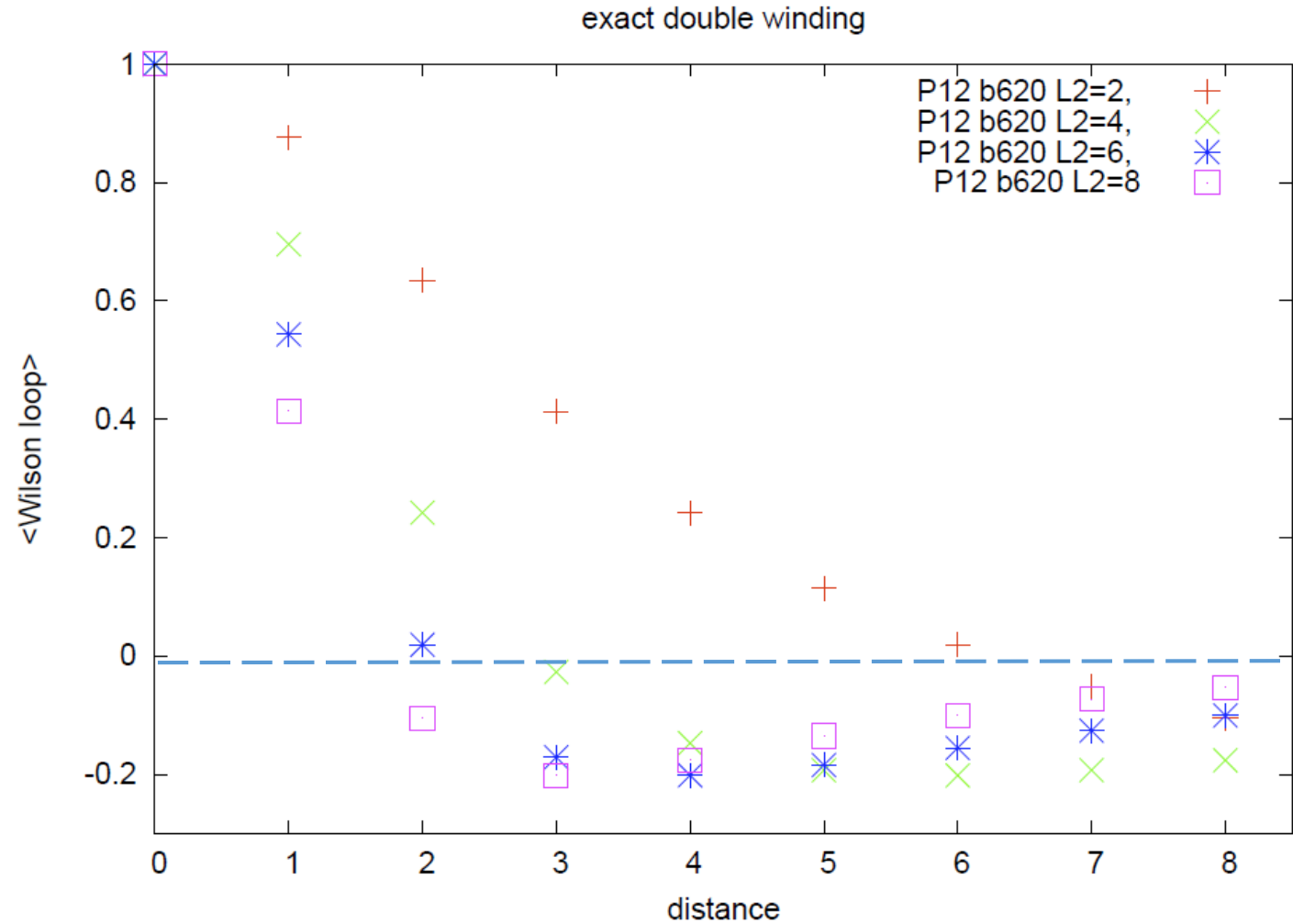
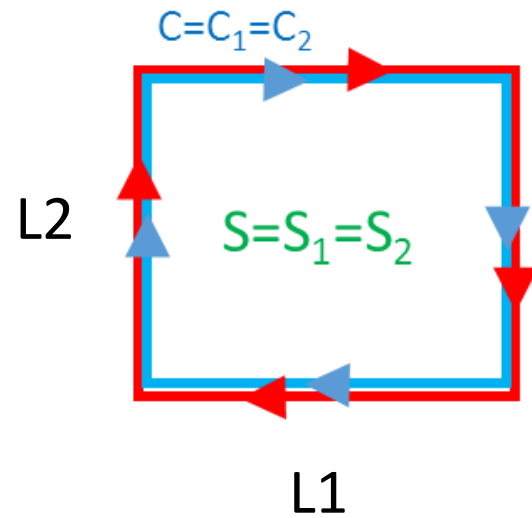
L=8



Single-Winding Wilson loop : SU(3)



Double-winding Wilson loop with the identical loop ($C_1=C_2$)

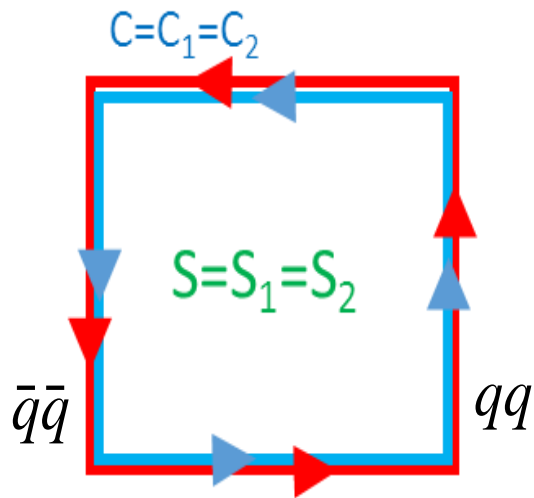


Summary and discussion

- We have investigated the double-winding Wilson loop average for $SU(N)$ Yang-Mills theory by using the strong coupling expansion and lattice simulation.
- By using the strong coupling expansion we obtain:
 - For $SU(2)$ case, the difference of area behavior
 - For $SU(N)$ ($N > 2$), **neither difference of area behavior nor sum of the area behavior**. In the large N_c limit, sum of the area behavior is expected.
- By numerical simulation, we confirm the result of the string coupling expansion for large area of S^1 .

- These results are consistent with another analysis: (talk by Matsudo)

Double Winding Wilson loops and high dimensional representation



$$SU(2) \text{ case : } 2 \otimes 2 = 2 \otimes 2^* = 1 \oplus 3$$

$$\langle W(C \times C) \rangle = -\frac{1}{2} + \frac{3}{2} \langle W_{\text{adj}}(C) \rangle$$

$$SU(3) \text{ case : } 3 \otimes 3 = 3^* \oplus 6$$

$$\langle W(C \times C) \rangle = -\langle W(C)_{[0,1]} \rangle + 2 \langle W(C)_{[2,0]} \rangle$$

$$SU(N) \text{ case : } N \otimes N = \left(\frac{N(N-1)}{2} \right)_A \oplus \left(\frac{N(N+1)}{2} \right)_S$$

$$\langle W(C \times C) \rangle = -\frac{N-1}{2} \langle W(C)_{[0,1,\dots,0]} \rangle + \frac{N+1}{2} \langle W(C)_{[2,0,\dots,0]} \rangle$$

See Matsudo's talk

Outlook

- Relation to N-ality
- Relation to the non-Abelian stokes theorem for quarks in the higher dimensional representation.
- Understanding in view of the dual-superconductivity

