Numerical study of the $2 + 1d$ Thirring model with U($2N$)-invariant fermions

Simon Hands[1],[*]

[1]*Department of Physics, College of Science,*
Swansea University, Singleton Park, Swansea SA2 8PP, United Kingdom

Abstract. In 2+1 dimensions the global U($2N$) symmetry associated with massless Dirac fermions is broken to U(N)⊗U(N) by a parity-invariant mass. I will show how to adapt the domain wall formulation to recover the U($2N$)-invariant limit in interacting fermion models as the domain wall separation is increased. In particular, I will focus on the issue of potential dynamical mass generation in the Thirring model, postulated to take place for N less than some critical N_c. I will present results of simulations of the model using both HMC ($N = 2$) and RHMC ($N = 1$) algorithms, and show that the outcome is very different from previous numerical studies of the model made with staggered fermions, where the corresponding pattern of symmetry breaking is distinct.

1 Introduction

Relativistic fermions moving in 2+1 spacetime dimensions form the basis of some facscinating quantum field theories, whose principal applications mainly fall in the domain of the condensed matter physics of layered systems. For example, fermion degrees of freedom described by a Dirac equation have been invoked and studied in models of the underdoped regime of d-wave superconductors [1, 2], where they arise as excitations at the nodes of the gap function $\Delta(\vec{k})$; in models of spin-liquid behaviour in Heisenberg antiferromagnets [3, 4], where the infrared behaviour of compact QED$_3$ remains an open issue; as surface states of three-dimensional topological insulators; and of course as low-energy electronic excitations in graphene [5].

What all these models share in common is their use of four-component *reducible* spinor representations for the fermi fields. For non-interacting fermions the action in Euclidian metric is

$$S = \int d^3x \, \bar{\psi}(\gamma_\mu \partial_\mu)\psi + m\bar{\psi}\psi; \quad \mu = 0, 1, 2; \tag{1}$$

a key point is the mass term proportional to m is hermitian and invariant under the parity inversion $x_\mu \mapsto -x_\mu$. For $m = 0$ S has a global U(2) invariance generated by

$$\psi \mapsto e^{i\alpha}\psi, \ \bar{\psi} \mapsto \bar{\psi}e^{-i\alpha}; \qquad \psi \mapsto e^{\alpha\gamma_3\gamma_5}\psi, \ \bar{\psi} \mapsto \bar{\psi}e^{-\alpha\gamma_3\gamma_5}; \tag{2}$$

$$\psi \mapsto e^{i\alpha\gamma_3}\psi, \ \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_3}; \quad \psi \mapsto e^{i\alpha\gamma_5}\psi, \ \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}. \tag{3}$$

For $m \neq 0$ the γ_3 and γ_5 rotations (3) are no longer symmetries and the general pattern of breaking is thus U(2N) →U(N)⊗U(N), where we generalise to N degenerate flavors.

Because there is no chiral anomaly in 2+1d, it is possible to perform a change of variables in the path integral to identify two further "twisted" mass terms which, though antihermitian, are physically equivalent to the $m\bar{\psi}\psi$ of (1):

$$im_3\bar{\psi}\gamma_3\psi; \quad im_5\bar{\psi}\gamma_5\psi. \tag{4}$$

The "Haldane" mass term $m_{35}\bar{\psi}\gamma_3\gamma_5\psi$ is not equivalent because it changes sign under parity, and will not be considered further.

2 The Thirring model in 2+1d

A model of particular interest is the Thirring model, which has a contact interaction between conserved fermion currents. Its Lagrangian density reads

$$\mathcal{L} = \bar{\psi}_i(\slashed{\partial} + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2; \quad i = 1, \ldots, N. \tag{5}$$

Equivalently, its dynamics is captured by a bosonised action via the introduction of an auxiliary vector field A_μ:

$$\mathcal{L}_A = \bar{\psi}_i(\slashed{\partial} + i\frac{g}{\sqrt{N}}A_\mu\gamma_\mu + m)\psi_i + \frac{1}{2}A_\mu^2. \tag{6}$$

The Thirring model is arguably the simplest interacting theory of fermions requiring a computational approach. The coupling g^2 has dimension $2 - d$, where d is the dimension of spacetime, so a naive expansion in powers of g^2 is non-renormalisable for $d > 2$. However, things look different after resummation. First introduce an additional Stückelberg scalar φ so the bosonic term becomes $\frac{1}{2}(A_\mu - \partial_\mu\varphi)^2$, to identify a hidden local symmetry [6]

$$\psi \mapsto e^{i\alpha}\psi; \quad A_\mu \mapsto A_\mu + \partial_\mu\alpha; \quad \varphi \mapsto \varphi + \alpha. \tag{7}$$

This point of view strongly suggests the identification of A_μ as an abelian gauge field; the original Thirring model (6) corresponds to a unitary gauge $\varphi = 0$. In Feynman gauge, to leading order in $1/N$ the resummed vector propagator is now of the form $\langle A_\mu(k)A_\nu(-k)\rangle \propto \delta_{\mu\nu}/k^{d-2}$. An expansion in powers of $1/N$ may now be developed by analogy with QED$_3$, and is exactly renormalisable for $2 < d < 4$ [7].

The outstanding theoretical issue is whether, for g^2 sufficiently large and N sufficiently small, there is a symmetry-breaking transition leading to formation of a bilinear condensate $\langle\bar{\psi}\psi\rangle \neq 0$ accompanied by dynamical fermion mass generation. It is of particular interest to identify the critical N_c above which symmetry breaking does not occur even in the strong coupling limit. The transitions at $g_c^2(N < N_c)$ then potentially define a series of distinct quantum critical points (QCPs). One early prediction, using strong-coupling Schwinger-Dyson equations in the ladder approximation, found $N_c \simeq 4.32$ [6].

3 The Thirring model with staggered fermions

The Thirring model has been studied by numerical simulations using staggered fermions, with action [8]

$$S_{stag} = \frac{1}{2}\sum_{x\mu i}\left[\bar{\chi}_x^i\eta_{\mu x}(1 + iA_{\mu x})\chi_{x+\hat{\mu}}^i - \text{h.c}\right] + m\sum_{xi}\bar{\chi}_x^i\chi_x^i + \frac{N}{4g^2}A_{\mu x}^2. \tag{8}$$

Eq. (8) is not unique, but has the feature that the linear coupling of the auxiliary precludes higher-point interactions between fermions once it is integrated over. The action (8) has a global U(N)⊗U(N) symmetry broken to U(N) by either explicit or spontaneous mass generation. In a weakly coupled long-wavelength limit a $U(2N_f)$ symmetry is recovered with $N_f = 2N$ [9]; however the putative QCP is not weakly coupled, so this conventional wisdom must be questioned.
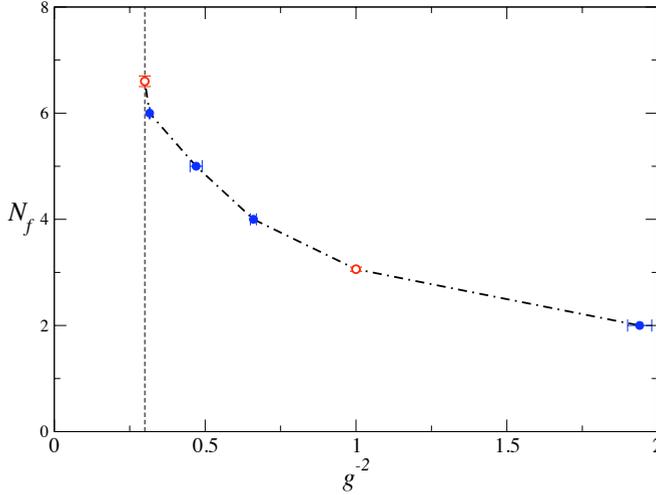

Figure 1. Phase diagram for the Thirring model with staggered fermions. A bilinear condensate is spontaneously generated in the region at lower left. The vertical dashed line denotes the effective strong coupling limit.

The action (8) has been studied for $N_f \in [2, 6]$ [8, 10, 11] and QCPs have been identified and characterised. Other studies with $N_f = 2$ have found compatible results [12, 13]. A more recent study which plausibly identifies the strong-coupling limit reported $N_{fc} = 6.6(1)$, $\delta(N_{fc}) \approx 7$ [14]. The results are summarised in Fig. 1. The "chiral" symmetry is indeed broken for small N_f and large g^2, and the exponent δ defined by the critical scaling $\langle \bar{\chi}\chi \rangle|_{g_c^2} \propto m^{\frac{1}{\delta}}$ is very sensitive to N_f. Other exponents can be estimated using hyperscaling. These results are in qualitative but not quantitative agreement with the Schwinger-Dyson approach, which predicts $\delta(N_{fc}) = 1$. A non-covariant form of (8) has been used to model the semimetal-insulator transition in graphene, finding $N_{fc} \approx 5$ [15] suggesting that a Mott insulating phase is possible for the $N_f = 2$ appropriate for monolayer graphene [16].

The staggered Thirring model thus exhibits a non-trivial phase diagram, with a sequence of QCPs with an N-dependence quite distinct from those of the theoretically much better-understood Gross-Neveu (GN) model. However, recent simulations of $N_f = 2$ with a fermion bag algorithm which permits study directly in the massless limit, have found compatible exponents for the QCP [13, 17]:

$$
\begin{aligned}
\nu &= 0.85(1); \quad \eta = 0.65(1); \quad \eta_\psi = 0.37(1); \quad \text{Thirring} \\
\nu &= 0.849(8); \quad \eta = 0.633(8); \quad \eta_\psi = 0.373(3); \quad \text{GN}
\end{aligned}
\tag{9}
$$

The large-N GN values are $\nu = \eta = 1$. These results are troubling: from the perspective of the large-N expansion using bosonised actions the models should be distinct, whereas (9) suggest rather they lie in the same RG basin of attraction. Indeed, when written purely in terms of four-point interactions between staggered fields spread over elementary cubes, the only difference between the models is an extra body-diagonal coupling in the GN case [17].

In this study we examine the possibility that staggered fermions do not reproduce the expected physics because of a failure to capture the correct continuum global symmetries near a QCP. A similar insight has been offered by the Jena group [18].

4 Domain wall fermions in 2+1d

The physical idea of domain wall fermions (DWF) is that fermions $\Psi, \bar{\Psi}$ are allowed to propagate along an extra fictitious dimension of extent L_s with open boundary conditions. In 2+1+1d this propagation is governed by an operator $\sim \partial_3 \gamma_3$. As $L_s \to \infty$ zero modes of D_{DWF} localised on the domain walls at either end become $\pm$ eigenmodes of γ_3, and physical fields in the target 2+1d space identified via

$$\psi(x) = P_- \Psi(x, 1) + P_+ \Psi(x, L_s); \quad \bar{\psi}(x) = \bar{\Psi}(x, L_s) P_- + \bar{\Psi}(x, 1) P_+, \tag{10}$$

with $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$. The walls are then coupled with a term proportional to the explicit massgap m.

However, the emergence of the U(2N) symmetry outlined in Sec. 1 is not manifest, because while the wall modes are eigenmodes of γ_3, the continuum symmetry (2,3) demands equivalence under rotations generated by both γ_3 *and* γ_5. Another way of seeing this is that the twisted mass terms (4) should yield identical physics, eg. the strength of the corresponding bilinear condensate, as $L_s \to \infty$. This requirement is apparently non-trivial, since while m and m_3 couple Ψ and $\bar{\Psi}$ fields on opposite walls, m_5 couples fields on the *same* wall.

The recovery of U(2N) symmetry as $L_s \to \infty$ was demonstrated numerically in a study of quenched non-compact QED$_3$ on $24^3 \times L_s$ systems, for a range of couplings β [19]. First define the principal residual Δ via the imaginary part of the twisted condensate:

$$\Im[i\langle \bar{\Psi}(1)\gamma_3 \Psi(L_s)\rangle] = -\Im[i\langle \bar{\Psi}(L_s)\gamma_3 \Psi(1)\rangle] \equiv \Delta(L_s). \tag{11}$$

The difference between the various condensates and their value in the large-L_s limit is then specified in terms of secondary residuals $\varepsilon_i(L_s)$ via

$$\begin{aligned}
\langle \bar{\psi}\psi \rangle_{L_s} &= i\langle \bar{\psi}\gamma_3\psi \rangle_{L_s \to \infty} + 2\Delta(L_s) + 2\varepsilon_h(L_s); \\
i\langle \bar{\psi}\gamma_3\psi \rangle_{L_s} &= i\langle \bar{\psi}\gamma_3\psi \rangle_{L_s \to \infty} + 2\varepsilon_3(L_s); \\
i\langle \bar{\psi}\gamma_5\psi \rangle_{L_s} &= i\langle \bar{\psi}\gamma_3\psi \rangle_{L_s \to \infty} + 2\varepsilon_5(L_s).
\end{aligned} \tag{12}$$

Empirically, as shown in Fig. 2, the residuals Δ and ε_i decay exponentially with L_s, with a clear hierarchy $\Delta \gg \varepsilon_h \gg \varepsilon_3 \equiv \varepsilon_5$. This is strong evidence for the ultimate recovery of U(2N) (the convergence rate is strongly dependent on both β and system volume), and moreover suggests the optimal simulation strategy is to focus on the twisted condensate $i\langle \bar{\psi}\gamma_3\psi \rangle$ for which finite-L_s corrections are minimal. The equivalence of γ_3 and γ_5 condensates at finite L_s and their superior convergence to the large-L_s limit was shown analytically in [20], where the convergence to fermions obeying 2+1d Ginsparg-Wilson relations was also demonstrated. Exponential improvement of convergence was shown in the large-N limit of the GN model in [21]. The benefits of a twisted mass term for improved recovery of U(2N) symmetry were also observed in a study of non-compact QED$_3$ using Wilson fermions in [22].

5 The Thirring model with domain wall fermions

Even after settling on DWF, we still encounter some remaining formulational issues. Just as for the staggered model, we have chosen a linear interaction between the fermion current and a 2+1d vector
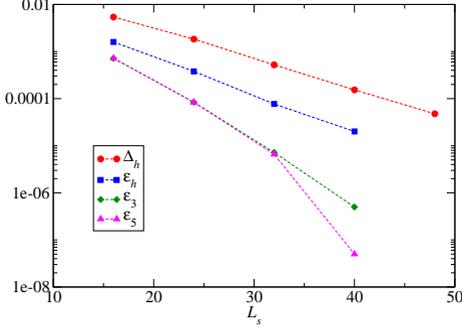
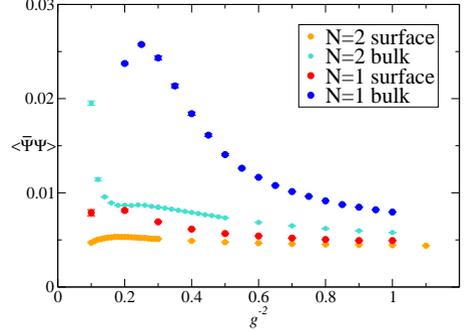**Figure 2.** Residuals as a function of L_s in quenched non-compact QED$_3$.

Figure 3. Bilinear condensate vs. g^{-2} for both surface and bulk Thirring models.

auxiliary field A_μ defined on the lattice links. The simplest approach to formulating the Thirring model is to restrict the interaction to the physical fields (10) defined on the domain walls at $s = 1, L_s$. This follows the treatment of the GN model with DWF developed in [23]. It has the technical advantage that the Pauli-Villars fields required to cancel bulk mode contributions to the fermion determinant do not couple to A_μ, and hence can be safely excluded from the simulation, which brings significant cost savings. In what follows this approach will be referred to as the *Surface* model.

However, following the discussion below eq. (7) suggesting the strong similarity of A_μ with an abelian gauge field, we also consider a *Bulk* formulation in which Ψ, $\bar\Psi$ interact with a "static" field (ie. $\partial_3 A_\mu = 0$) throughout the bulk:

$$S = \bar\Psi \mathcal{D}\Psi = \bar\Psi D_W \Psi + \bar\Psi D_3 \Psi + m_i S_i, \tag{13}$$

with

$$D_W = \gamma_\mu D_\mu - (\hat{D}^2 + M); \quad D_3 = \gamma_3 \partial_3 - \hat{\partial}_3^2; \tag{14}$$

and $m_i S_i$ is the explicit mass term defined only on the walls. Here

$$
\begin{aligned}
D_{\mu xy} &= \frac{1}{2}\left[(1 + iA_{\mu x})\delta_{x+\hat\mu,y} - (1 - iA_{\mu x-\hat\mu})\delta_{x-\hat\mu,y}\right], \\
\hat{D}_{xy}^2 &= \frac{1}{2}\sum_\mu \left[(1 + iA_{\mu x})\delta_{x+\hat\mu,y} + (1 - iA_{\mu x-\hat\mu})\delta_{x-\hat\mu,y} - 2\delta_{xy}\right],
\end{aligned}
\tag{15}
$$

and M is the domain wall height, here set equal to 1. $\partial_3, \hat{\partial}_3^2$ are defined similarly using finite differences in the 3 direction, respecting the open boundary conditions, and with no coupling to the auxiliary. These definitions imply the following properties:

$$[\partial_3, D_\mu] = [\partial_3, \hat{D}^2] = 0 \quad \text{but} \quad [\partial_3, \hat{\partial}_3^2] \neq 0 \quad \text{on domain walls.} \tag{16}$$

The action (13) may be simulated using the HMC algorithm for N even; however the failure of the third commutator in (16) to vanish everywhere is an obstruction to proving $\det\mathcal{D}$ is positive definite, so $N = 1$ is simulated using the RHMC algorithm with functional measure $\det(\mathcal{D}^\dagger\mathcal{D})^{\frac{1}{2}}$.
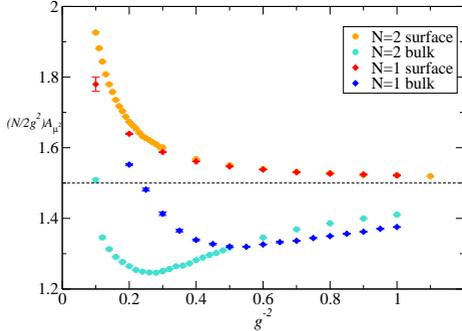
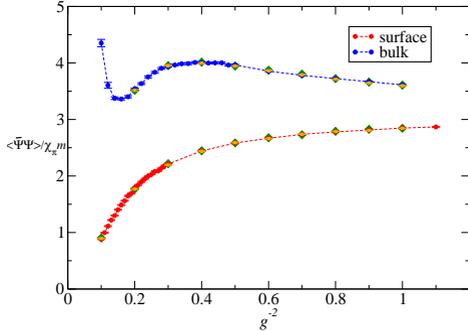**Figure 4.** Auxiliary action vs. g^{-2}.

Figure 5. Test of the axial Ward Identity.

6 Numerical results

We have studied both surface and bulk formulations of the Thirring model in the coupling range $ag^{-2} \in [0.1, 1.0]$ with first $N = 2$ [21] and now $N = 1$. The RHMC algorithm used 25 partial fractions to estimate fractional powers of $\mathcal{D}^\dagger\mathcal{D}$. Most results are obtained on $12^3 \times 16$ ($N = 2$) or $12^3 \times 8$ ($N = 1$). An exploration of volume and finite-L_s effects for $N = 2$ was presented in [21]. Summary results for the bilinear condensate $i\langle\bar{\psi}\gamma_3\psi\rangle$ with $m_3 a = 0.01$ are shown in Fig. 3. The bulk model shows signifcantly enhanced pairing for $g^{-2} \lesssim 0.5$ compared to the surface model, and as might be anticipated pairing is greater for $N = 1$ than $N = 2$. This trend continues until a maximum at $ag^{-2} \approx 0.2$. In previous work with staggered fermions this has been identified with the effective location of the continuum strong coupling limit [14], since for stronger lattice couplings there is a breakdown of reflection positivity [8]. We are thus confident that the range of couplings explored includes the strong coupling limit.

Fig. 4 shows the auxiliary action over the same coupling range, compared with the free field value $\frac{d}{2}$. The difference between surface and bulk models is apparently very striking, but it should be borne in mind that this is really a comparison of UV properties of two different regularisations of ostensibly the same continuum field theory. Once again, there is tentative evidence for a change of behaviour of the $N = 1$ bulk model at $ag^{-2} \approx 0.5$.

Fig. 5 shows the ratio $i\langle\bar{\psi}\gamma_3\psi\rangle/m_3\chi_\pi$ obtained for $N = 2$, with the transverse susceptibility defined

$$\chi_\pi = N \sum_x \langle\bar{\psi}\gamma_5\psi(0)\bar{\psi}\gamma_5\psi(x)\rangle. \tag{17}$$

For a theory where the $\psi, \bar{\psi}$ fields respect U(2N) symmetry, the 2+1d generalisation of the axial Ward identity predicts the ratio to be unity. Fig. 5 shows that this requirement is far from being met, and that further work is needed to understand and calibrate the identification of the physical fields via relations such as (10). Again, the disparity between bulk and surface models is striking, with neither being obviously preferred.

Finally we consider whether U(2N) is spontaneously broken at strong coupling. In [21] the bilinear condensate for $N = 2$ was examined as a function of bare mass m across a range of couplings, and in every case a linear scaling $\langle\bar{\psi}\psi\rangle \propto m$ was found, indicative of U(2N) symmetry being manifest as $m \to 0$. We now extend this study to $N = 1$. Figs. 6,7 show $\langle\bar{\psi}\psi(m)\rangle$ for $ma = 0.01, \ldots, 0.05$;
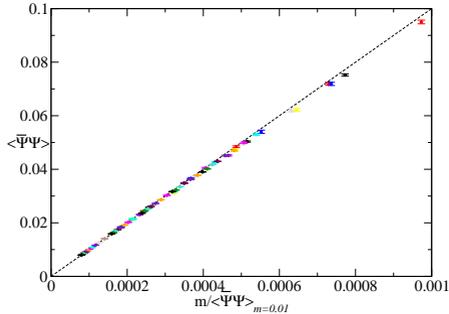
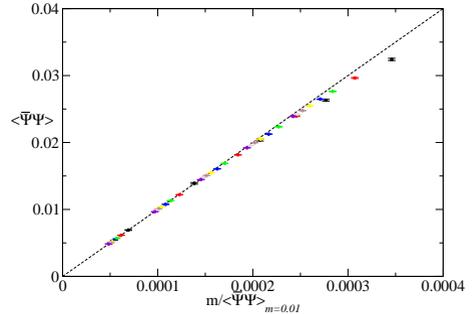**Figure 6.** Data collapse for the $N = 1$ bulk model...   **Figure 7.** ... and for the surface model.

with a trivial rescaling implemented by choosing the abscissa as $m/\langle\bar{\psi}\psi(am = 0.01)\rangle$, data from the entire range of couplings studied collapses onto a near linear curve, for both bulk (left) and surface (right) models. There is no evidence for any singular behaviour associated with a symmetry-breaking phase transition, and it seems safe to conclude $\lim_{m\to 0}\langle\bar{\psi}\psi\rangle = 0$. Assuming this picture persists in the large volume and $L_s \to \infty$ limits, this provides strong evidence that the critical flavor number in the U(2N)-symmetric Thirring model is constrained by

$$N_c < 1. \tag{18}$$

7 Summary and outlook

It has been shown that it is feasible to use DWF to study U(2N)-symmetric fermions in 2+1d, and that use of a twisted mass term $\sim im_3\bar{\psi}\gamma_3\psi$ optimises the recovery of the symmetry as $L_s \to \infty$. A study of the Thirring model at strong fermion self-couplings then shows that DWF capture a very different physics to that described by staggered fermions, which are governed by a different global symmetry away from the weak-coupling long-wavelength limit. While it is still not possible to settle on the preferred formulation of the strongly-coupled Thirring model, both the bulk and surface versions presented here are in agreement that the critical flavor number $N_c < 1$. Fortunately this is compatible with the results obtained using a distinct U(2N)-symmetric approach involving the SLAC derivative [24], also presented at this conference [25]. Recent studies of U(2N)-symmetric QED$_3$, an asymptotically-free theory, have also concluded $N_c < 1$ [22, 26]. It is noteworthy that a disparity between DWF and staggered fermions in a very different physical context, namely near a conformal fixed point in a 3+1d non-abelian gauge theory, has also been reported [27].

In future work it will obviously necessary to study the effects of $L_s \to \infty$, $V \to \infty$, and of varying the domain wall height M. It will also be valuable to study the locality properies of the corresponding 2+1d overlap operator, furnishing a non-trivial test of the DWF approach in a new physical context. Another question to ponder is what does chiral symmetry breaking actually look like for 2+1+1d DWF? We are currently investigating this issue by quenching the Thirring model. Finally, functional renormalisation group studies indicate that in the hunt for a QCP it may be interesting to include a U(2N)-invariant Haldane interaction $-\tilde{g}^2(\bar{\psi}\gamma_3\gamma_5\psi)^2$ [28]; the control offered by DWF will make this a straightforward exercise.

References

[1] Z. Tesanovic, O. Vafek, M. Franz, Phys. Rev. **B65**, 180511 (2002)

[2] I.F. Herbut, Phys. Rev. **B66**, 094504 (2002), `cond-mat/0202491`

[3] X.G. Wen, Phys. Rev. **B65**, 165113 (2002)

[4] W. Rantner, X.G. Wen, Phys. Rev. **B66**, 144501 (2002)

[5] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, A.K. Geim, Rev. Mod. Phys. **81**, 109 (2009)

[6] T. Itoh, Y. Kim, M. Sugiura, K. Yamawaki, Prog. Theor. Phys. **93**, 417 (1995), `hep-th/9411201`

[7] S. Hands, Phys. Rev. **D51**, 5816 (1995), `hep-th/9411016`

[8] L. Del Debbio, S.J. Hands, J.C. Mehegan (UKQCD), Nucl. Phys. **B502**, 269 (1997), `hep-lat/9701016`

[9] C. Burden, A.N. Burkitt, Europhys. Lett. **3**, 545 (1987)

[10] L. Del Debbio, S.J. Hands, Nucl. Phys. **B552**, 339 (1999), `hep-lat/9902014`

[11] S. Hands, B. Lucini, Phys. Lett. **B461**, 263 (1999), `hep-lat/9906008`

[12] E. Focht, J. Jersak, J. Paul, Phys. Rev. **D53**, 4616 (1996), `hep-lat/9511005`

[13] S. Chandrasekharan, A. Li, Phys. Rev. Lett. **108**, 140404 (2012), `1111.7204`

[14] S. Christofi, S. Hands, C. Strouthos, Phys. Rev. **D75**, 101701 (2007), `hep-lat/0701016`

[15] S. Hands, C. Strouthos, Phys. Rev. **B78**, 165423 (2008), `0806.4877`

[16] W. Armour, S. Hands, C. Strouthos, Phys. Rev. **B81**, 125105 (2010), `0910.5646`

[17] S. Chandrasekharan, A. Li, Phys. Rev. **D88**, 021701 (2013), `1304.7761`

[18] D. Schmidt, B. Wellegehausen, A. Wipf, PoS **LATTICE2015**, 050 (2016), `1511.00522`

[19] S. Hands, JHEP **09**, 047 (2015), `1507.07717`

[20] S. Hands, Phys. Lett. **B754**, 264 (2016), `1512.05885`

[21] S. Hands, JHEP **11**, 015 (2016), `1610.04394`

[22] N. Karthik, R. Narayanan, Phys. Rev. **D93**, 045020 (2016), `1512.02993`

[23] P. Vranas, I. Tziligakis, J.B. Kogut, Phys. Rev. **D62**, 054507 (2000), `hep-lat/9905018`

[24] B.H. Wellegehausen, D. Schmidt, A. Wipf (2017), `1708.01160`

[25] B. Wellegehausen, A. Wipf, D. Schmidt, *Four-Fermi-Theories in 3 Dimensions: Critical flavour number of the massless Thirring model*, in *Proceedings, 35th International Symposium on Lattice Field Theory (Lattice2017): Granada, Spain*, to appear in EPJ Web Conf., `1710.01000`

[26] N. Karthik, R. Narayanan, Phys. Rev. **D94**, 065026 (2016), `1606.04109`

[27] A. Hasenfratz, C. Rebbi, O. Wetzel, *Testing Universality at a Conformal Fixed Point*, in *Proceedings, 35th International Symposium on Lattice Field Theory (Lattice2017): Granada, Spain*, to appear in EPJ Web Conf., `1708.03385`

[28] F. Gehring, H. Gies, L. Janssen, Phys. Rev. **D92**, 085046 (2015), `1506.07570`